Math 107: Calculus II, Spring 2008: Practice Final II Solutions

1. (a) Separating the variables gives us:

$$\int \frac{1}{y^2}\, dy = \int -3x^2\, dx$$

Integrating, we get

$$\frac{-1}{y} = -x^3 + c$$

and so

$$y = \frac{1}{x^3 - c}.$$

We divided by y^2 so $y = 0$ is a 'missing' solution and so the general solution is

$$\boxed{y = \frac{1}{x^3 - c} \text{ or } y = 0}$$

(b) To get $y(0) = 1/5$, we substitute in $x = 0$ and $y = 1/5$ to get

$$\frac{1}{5} = \frac{1}{-c}$$

which tells us that $c = -5$. Therefore, the specific solution we are looking for is

$$\boxed{y = \frac{1}{x^3 + 5}}$$

(c) Looking at the general solution, it is clear that the solution $\mathbf{y = 0}$ has $y(0) = 0$ and so this is the specific solution we are looking for.

2. (a) We first write this system of equations using matrices:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

We then have to find the eigenvalues and eigenvectors of the 2×2-matrix here. To find the eigenvalues we look at

$$\det\begin{pmatrix} 2-k & 4 \\ -1 & -3-k \end{pmatrix} = (2-k)(-3-k) + 4 = k^2 + k - 2$$

and set it equal to zero. This can be written

$$(k+2)(k-1) = 0$$

and so the eigenvalues are $k = -2$ and $k = 1$.

We now find the eigenvectors. To find an eigenvector for $k = -2$, we have to solve

$$\begin{pmatrix} 2-(-2) & 4 \\ -1 & -3-(-2) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This comes out as:

$$4u + 4v = 0; \quad -u - v = 0.$$

We can choose any solution that is not u, v both zero, so for example, we can take

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This is therefore an eigenvector with eigenvalue -2.
To find an eigenvector for $k = 1$, we have to solve

$$\begin{pmatrix} 2-1 & 4 \\ -1 & -3-1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which comes out as

$$u + 4v = 0; \quad -u - 4v = 0$$

so as solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

Putting it all together tells us that the general solution to the original system of differential equations is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

or

$$\boxed{x(t) = c_1 e^{-2t} + 4c_1 e^{t}, \quad y(t) = -c_1 e^{-2t} - c_2 e^{t}}$$

(b) The equilibrium solution is the constant solution $x = 0, y = 0$. This is **unstable** because at least one of the eigenvalues is positive.

3. (a) The equilibrium solutions are $\mathbf{y = 0, 1, 2}$.

(b) If $g(y) = y(y-2)(y-1) = y^3 - 3y^2 + 2y$, then $g'(y) = 3y^2 - 6y + 2$. Then:

- $g'(0) = 2$ which is positive so $y = 0$ is **unstable**;
- $g'(1) = -1$ which is negative so $y = 1$ is **stable**;
- $g'(2) = 2$ which is positive so $y = 2$ is **unstable**.

(c) See separate sheet of graphs.

(d) Ditto.

4. (a) The partial derivatives of $f(x,y)$ are

$$\frac{\partial f}{\partial x} = 2\cos(2x+y)e^{\sin(2x+y)}, \quad \frac{\partial f}{\partial y} = \cos(2x+y)e^{\sin(2x+y)}.$$

Therefore, the directional derivative at $(0,0)$ in the direction $\begin{pmatrix}3\\1\end{pmatrix}$ is

$$\frac{\frac{\partial f}{\partial x}(0,0)\times 3 + \frac{\partial f}{\partial y}(0,0)\times 1}{\sqrt{3^2+1^2}} = \frac{2\times 3 + 1\times 1}{\sqrt{10}} = \boxed{\frac{7}{\sqrt{10}}}$$

(b) The directional derivative is at a maximum in the direction of the gradient vector

$$\nabla f(0,0) = \boxed{\begin{pmatrix}2\\1\end{pmatrix}}$$

(c) The value of the maximum directional derivative is equal to the magnitude of the gradient vector:

$$|\nabla f(0,0)| = \sqrt{2^2+1^2} = \boxed{\sqrt{5}}$$

(d) The linear approximation to f at $(0,0)$ is

$$f(x,y) \simeq f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0)$$

which is

$$f(x,y) \simeq 1 + 2x + y.$$

5. (a) See extra page of graphs.

(b) You would expect $\frac{\partial f}{\partial x}(3,4)$ to be **positive** and $\frac{\partial f}{\partial y}(3,4)$ to be **negative**.

(c) See extra page of graphs.

(d) Ditto.

6. (a) The partial derivatives of $f(x,y)$ are

$$\frac{\partial f}{\partial x} = 2xy - 2x, \quad \frac{\partial f}{\partial y} = x^2 - 9.$$

To find the critical points, we set these equal to zero. Therefore $x^2 - 9 = 0$ which gives $x = \pm 3$. The first equation is then

$$2x(y-1) = 0$$

which, since we already know $x \neq 0$, gives $y = 1$. Therefore the critical points are $(3,1)$ and $(-3,1)$.

To classify these critical points as local max/mins or saddle points, we find the Hessian matrix at each one. The Hessian is

$$Hf = \begin{pmatrix} 2y-2 & 2x \\ 2x & 0 \end{pmatrix}.$$

At the critical point $(3,1)$ this is

$$\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}.$$

The determinant here is -36 which is negative and so $(3,1)$ is a **saddle point**. At $(-3,1)$, the Hessian is

$$\begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$$

which also has determinant -36 so $(-3,1)$ is also a **saddle point**.

(b) For any values of x and y we have $x^4 \geq 0$ and $y^4 \geq 0$, and hence $x^4 + y^4 \geq 0$. This means that $(0,0)$ is a global minimum for $g(x,y)$ since $g(x,y) \geq g(0,0)$ for all x, y. The function $g(x,y)$ does **not** have a global maximum since the function can be made arbitrarily large by making x and/or y large.

7. (a) Look at the possible outcomes for the first two days:

- RR: probability $3/5 \times 2/4 = 3/10$
- RB: probability $3/5 \times 2/4 = 3/10$
- BR: probability $2/5 \times 3/4 = 3/10$
- BB: probability $2/5 \times 1./4 = 1/10$

Therefore the probability of wearing a blue shirt on Tuesday is

$$P(RB) + P(BB) = 4/10 = \boxed{2/5}$$

(b) $P(A) = 3/5$ and $P(B) = 2/5$. $P(A \text{ and } B) = P(RB) = 3/10$ which is not equal to $P(A)P(B)$ and so A and B are **not independent**.

(c) We need to find the distribution of X:

- $P(X = 1) = 2/5$
- $P(X = 2) = P(RB) = 3/10$
- $P(X = 3) = P(RRB) = 3/5 \times 2/4 \times 2/3 = 12/60 = 1/5$
- $P(X = 4) = P(RRRB) = 3/5 \times 2/4 \times 1/3 \times 2/2 = 12/120 = 1/10$
- since there are only 3 red shirts, the latest I will wear my first blue shirt is Thursday.

We then get

$$EX = 1 \times 2/5 + 2 \times 3/10 + 3 \times 1/5 + 4 \times 1/10 = 20/10 = \boxed{2}$$

8. (a) The range of possible values of X is from 0 to 2.

   (b) Values between 1 and 2 have a higher likelihood of occuring than values between 0 and 1. You would therefore expect the expectation to be above the midpoint of the range, i.e. greater than 1.

   (c) The expectation of X is

$$EX = \int_0^2 x(3x^2/8)\,dx = \int_0^2 3x^3/8\,dx = \left[\frac{3x^4}{32}\right]_0^2 = \frac{48}{32} = \boxed{\frac{3}{2}}$$

   (d) The cdf of X is

$$F(x) = \begin{cases} 0 & \text{for } x < 0; \\ x^3/8 & \text{for } 0 \le x \le 2; \\ 1 & \text{for } x > 2. \end{cases}$$

9. (a) If X is the maximum annual temperature, then we want $P(X > 115)$ which is

$$P(Z > (115 - 105)/5) = P(Z > 2) = 1 - P(Z \le 2) = 1 - 0.9772 = \boxed{0.0228}$$

   (b) If Y is the number of times that the temperature rises above 115 in the five year period then Y has a binomial distribution with $n = 5$ and $p = 0.0228$. We want $P(Y = 2)$ which is

$$\binom{5}{2}(0.0228)^2(1 - 0.0228)^3 = \boxed{10(0.0228)^2(0.9772)^3}$$

   (c) If $\overline{X}$ is the average temperature over 100 years, then by the Central Limit Theorem, $\overline{X}$ is approximately a normal distribution with mean 105 and standard deviation $\sqrt{\text{Var}(X)/n} = \sqrt{25/100} = 5/10 = 0.5$. Then

$$\begin{aligned} P(\overline{X} > 105.5) &\simeq P(Z > (105.5 - 105)/0.5) \\ &= P(Z > 1) = 1 - P(Z \le 1) = 1 - 0.8413 \\ &= \boxed{0.1587} \end{aligned}$$

10. (a) The best estimate is

$$\hat{p} = \frac{10}{50} = 0.2.$$

   The standard error of this estimate is

$$\text{S.E.} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}} = \sqrt{\frac{0.2 \times 0.8}{49}} = \frac{0.4}{7}.$$

   Therefore, the confidence interval is

$$\hat{p} \pm 1.96 \times \text{S.E.} = 0.2 \pm 1.96 \times \frac{0.4}{7} = 0.2 \pm 0.28 \times 0.4 = 0.2 \pm 0.112.$$

   So the confidence interval for $P(\text{heads})$ for this coin is

$$\boxed{[0.088, 0.312]}$$

(b) This is the normal approximation to the binomial distribution (This is actually not the Central Limit Theorem as we saw it in class. Sorry for the confusion here - I will make it clearer on the exam.) This says that the number X of heads is approximately a normal distribution with mean $\mu = np = 15$ and standard deviation $\sigma = \sqrt{np(1-p)} = \sqrt{50 \times 0.3 \times 0.7} = \sqrt{10.5}$. With the histogram correction we want $P(X \leq 10.5)$ which is therefore equal to

$$\boxed{P(Z \leq (10.5 - 15)/\sqrt{10.5})}$$