Math 107: Calculus II, Spring 2008: Practice Final II

1. (a) Find the general solution to the separable differential equation

$$\frac{dy}{dx} = -3x^2y^2.$$

   (b) Find the solution to the equation from part (a) that satisfies $y(0) = 1/5$.

   (c) Find the solution to the equation from part (a) that satisfies $y(0) = 0$.

2. This question concerns the following linear system of differential equations:

$$\frac{dx}{dt} = 2x + 4y, \quad \frac{dy}{dt} = -x - 3y.$$

   (a) Find the general solution to this system of equations.

   (b) What is the equilibrium solution? Is it stable or unstable? (Explain how you know.)

3. (a) What are the equilibrium solutions to the autonomous differential equation

$$\frac{dy}{dt} = y(y-2)(y-1)$$

   (b) Use the first derivative test to say whether the equilibria from part (a) are stable, unstable or semi-stable.

   (c) Draw a sketch of the solutions to the differential equation from part (a).

   (d) Sketch the graph of a function $g(y)$ such that the differential equation

$$\frac{dy}{dt} = g(y)$$

   has a stable equilibrium at $y = -1$ and an unstable equilibrium at $y = 1$. (Note: part (d) is not connected to the other parts of this question.)

4. Let f be the function of two variables given by

$$f(x,y) = e^{\sin(2x+y)}.$$

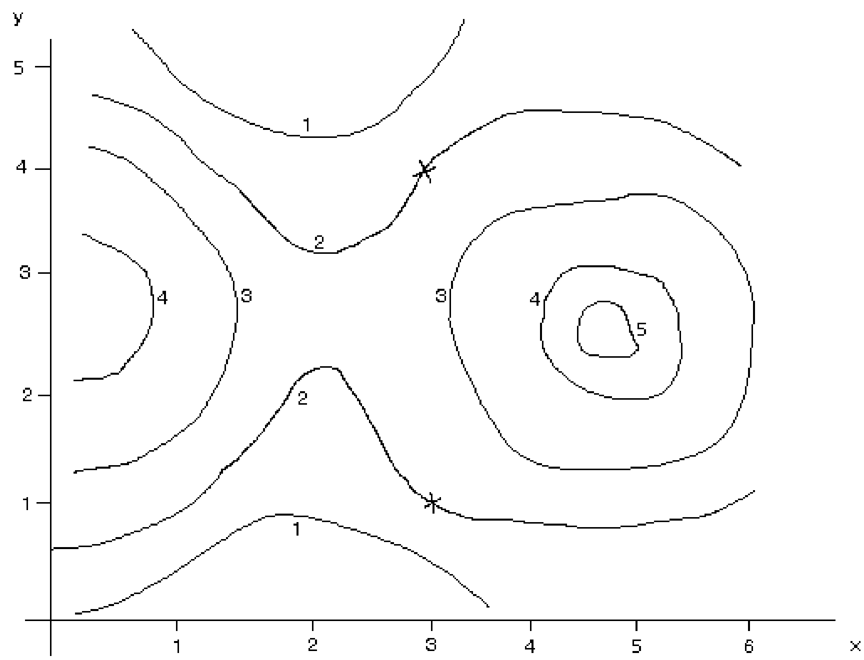
   (a) Find the directional derivative of f at the point $(0,0)$ in the direction $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

   (b) In which direction is the directional derivative of f at $(0,0)$ at a maximum?

   (c) What the value of the directional derivative of f at $(0,0)$ in the direction you found in part (b).

   (d) Find the linear approximation of f at the point $(0,0)$. (Note: your answer should be a linear function of x and y.)

5. The following diagram displays some of the level curves of the function $f(x, y)$. The number next to a curve denotes the value of f along that curve.



Answer the following question based on the information in this diagram. Note: the points marked with a cross are $(3, 1)$ and $(3, 4)$.

(a) Draw a graph of the $x = 2$ cross-section of the function f. (Make it as accurate as you can based on the information in the question.)

(b) For each of the partial derivatives $\dfrac{\partial f}{\partial x}(3, 4)$ and $\dfrac{\partial f}{\partial y}(3, 4)$, say if it is positive or negative?

(c) Draw a vector starting at the point $(3, 1)$ that represents the direction of the gradient vector $\nabla f(3, 1)$. (The length of the vector does not matter.)

(d) Mark on the above diagram where (approximately) you think any critical points of the function f may lie. Are these critical points local maxima, local minima or saddle points?

6. (a) Find all the critical points of the function

$$f(x, y) = x^2y - x^2 - 9y$$

and classify them as local maxima, local minima or saddle points.

(b) Show that the function $g(x, y) = x^4 + y^4$ has a global minimum at $(0, 0)$. Does g have a global maximum? Explain your answer.

7. On Monday, my closet contains 5 shirts: 3 red and 2 blue. I pick a shirt at random to wear. On Tuesday, I pick a shirt at random from the 4 that remain. On Wednesday I pick at random from the 3 that remain, and so on up to Friday. Answer the following questions.

   (a) Find the probability that I wear a blue shirt on Tuesday.

   (b) Decide if the following two events are independent or not:

   $A = \{\text{I wear a red shirt on Monday}\}$
   $B = \{\text{I wear a blue shirt on Tuesday}\}$

   (c) The random variable X is equal to the day on which I wear my *first* blue shirt (i.e. if I wear a blue shirt on Monday, then $X = 1$; if I wear red on Monday and blue on Tuesday, then $X = 2$, and so on.) Find the expectation of the random variable X.

8. The probability density function (pdf) of the continuous random variable X is given by

$$f(x) = \begin{cases} 3x^2/8 & \text{for } 0 \leq x \leq 2; \\ 0 & \text{otherwise.} \end{cases}$$

   (a) What is the range of possible values of X?

   (b) Explain (without calculating it) why you would predict that the expectation of X is greater than 1. (Hint: it will probably help to sketch the graph of the pdf.)

   (c) Calculate the expectation of X.

   (d) Find the cumulative distribution function (cdf) of X, that is the function $F(x) = P(X \leq x)$.

9. (a) The maximum annual temperature in Baltimore is normally distributed with mean 105 (degrees Fahrenheit) and standard deviation 5. Find the probability that in a given year, the temperature is greater than 115 degrees.

   (b) Find an expression for the probability that the temperature rises above 115 degrees in exactly two of the years in a five year period. (Your expression should be in a form that could easily be entered into a calculator. You do not need to find the probability.)

   (c) Use the Central Limit Theorem to get an approximation for the probability that the average maximum temperature over a hundred year period is greater than 105.5 degrees.

   (In parts (b) and (c), you should assume that the maximum temperatures for different years are independent.)

10. (a) I am trying to estimate the probability that a particular coin comes up heads. I flip the coin 50 times and observe 10 heads. Find a 95% confidence interval for $P(\text{heads})$. (Hint: the following calculation may be useful: $\frac{1.96}{7} = 0.28$.)

(b) Suppose that the coin in part (a) in fact has $P(\text{heads}) = 0.3$. Use the Central Limit Theorem (including the histogram correction) to write down an expression for the probability of getting less than or equal to 10 heads from 50 flips. Your answer should be of the form $P(Z \leq z)$ where Z has a standard normal distribution and z is some number. You should give an expression for z that could easily be entered into a calculator.