Math 107, Spring 2008: Midterm III Practice Exam Solutions

Questions

1. *A bag contains two red balls and three blue balls. I perform the following experiment. I pick one ball at random from the bag, and note its color. I then pick a second ball from the bag **without** replacing the first, and note its color.*

   (a) *List the possible outcomes of this experiment and find the probability of each outcome.*

   (b) *Let A stand for the event "second ball is red" and let B stand for the event "the two balls are the same color".*

   i. *Find $P(A)$.*

   ii. *Find $P(B)$.*

   iii. *Find $P(A \text{ and } B)$.*

   (c) *Are the events A and B independent? Explain how you know.*

   (a) The possible outcomes are BB, RB, BR, RR, where BR stands for "first ball blue, second ball red" and so on.

$$P(BB) = \frac{3}{5}\frac{2}{4} = \frac{3}{10}$$

$$P(RB) = \frac{2}{5}\frac{3}{5} = \frac{3}{10}$$

$$P(BR) = \frac{3}{5}\frac{2}{4} = \frac{3}{10}$$

$$P(RR) = \frac{2}{5}\frac{1}{4} = \frac{1}{10}$$

   (b) i. $P(A) = P(RR) + P(BR) = 4/10 = 2/5$.

   ii. $P(B) = P(RR) + P(BB) = 4/10 = 2/5$.

   iii. $P(A \text{ and } B) = P(RR) = 1/10$.

   (c) The two events A and B are **not independent** because

$$P(A \text{ and } B) \neq P(A)P(B).$$

2. *The discrete random variable X has the following distribution:*

   - $P(X = 1) = 3/5$
   - $P(X = 2) = 0$
   - $P(X = 3) = 1/5$
   - $P(X = 4) = 1/5$

(a) *Find the expectation and variance of X.*

(b) *Suppose we took 40 independent measurements of X. Find an estimate for the probability that the average of those measurements is greater than 2.4?*

(a) The expectation of X is given by

$$EX = 1 \times P(X = 1) + 3 \times P(X = 3) + 4 \times P(X = 4) = 1\frac{3}{5} + 3\frac{1}{5} + 4\frac{1}{5} = \frac{10}{5} = 2.$$

The variance of X is therefore

$$\begin{aligned}\mathrm{Var}(X) &= (1-2)^2 P(X = 1) + (3-2)^2 P(X = 3) + (4-2)^2 P(X = 4)\\ &= \frac{3}{5} + \frac{1}{5} + 4\frac{1}{5}\\ &= \frac{8}{5}.\end{aligned}$$

(b) By the Central Limit Theorem, the average of the 40 independent measurements is approximately a normal distribution with mean $\mu = EX = 2$ and standard deviation $\sigma = \sqrt{\mathrm{Var}(X)/n} = \sqrt{1.6/40} = \sqrt{0.04} = 0.2$. The probability that this average is greater than 2.4 is therefore equal to

$$P(Z \geq (2.4 - 2)/0.2) = P(Z \geq 2) = 1 - P(Z \leq 2) = 1 - 0.9772 = 0.0228.$$

3. *The continuous random variable X has cumulative distribution function (cdf) given by*

$$F(x) = \begin{cases} 0 & \textit{if } x < 0; \\ (3x^2 - x^3)/4 & \textit{if } 0 \leq x \leq 2; \\ 1 & \textit{if } x > 2. \end{cases}$$

(a) *Find the probability density function for X.*

(b) *Find $P(1 \leq X \leq 2)$.*

(c) *Find the expectation and variance of X.*

(a) The pdf is given by differentiating the cdf:

$$f(x) = \begin{cases} (6x - 3x^2)/4 & \text{if } 0 \leq x \leq 2; \\ 0 & \text{if } x < 0 \text{ or } x > 2. \end{cases}$$

(b) $P(1 \leq X \leq 2) = F(2) - F(1) = 1 - 0.5 = 0.5.$

(c) The expectation of X is

$$EX = \int_0^2 x(6x - 3x^2)/4\, dx = \int_0^2 (6x^2 - 3x^3)/4\, dx = [2x^3/4 - 3x^4/16]_0^2 = 4 - 3 = 1.$$

The variance is therefore

$$\begin{aligned}
\mathrm{Var}(X) &= \int_0^2 (x-1)^2(6x-3x^2)/4\,dx = \int_0^2 (x^2-2x+1)(6x-2x^2)/4\,dx \\
&= \int_0^2 (6x^3 - 12x^2 + 6x - 2x^4 + 4x^3 - 2x^2)/4\,dx \\
&= \int_0^2 (-2x^4 + 10x^3 - 14x^2 + 6x)/4\,dx \\
&= [-x^5/10 + 5x^4/8 - 7x^3/6 + 3x^2/4]_0^2 = (-32/10 + 10 - 28/3 + 3) \\
&= (-96 + 300 - 280 + 90)/30 \\
&= 7/15.
\end{aligned}$$

4. (a) *Suppose X has a normal distribution with mean 3 and standard deviation 4. Find:*

   i. $P(X \leq 5)$

   ii. $P(X \geq -2)$

(b) *Suppose Y is binomially distributed with 5 repetitions and probability of success 1/3. Find:*

   i. $P(Y = 5)$.

   ii. $P(Y \geq 1)$.

(a) i. $P(X \leq 5) = P(Z \leq (5-3)/4) = P(Z \leq 0.5) = 0.6915$.

ii. $P(X \geq -2) = P(Z \geq (-2-3)/4) = P(Z \geq -1.25) = 1 - P(Z \leq -1.25) = 1 - (1 - P(Z \leq 1.25)) = P(Z \leq 1.25) = 0.8944$.

(b) i. By the formula for binomial probabilities:

$$P(Y=5) = \binom{5}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0.$$

The binomial coefficient here is equal to 1, so we have

$$P(Y=5) = \left(\frac{1}{3}\right)^5 = \frac{1}{243}.$$

ii. We have $P(Y \geq 1) = 1 - P(Y = 0)$ which is equal to

$$1 - \binom{5}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^5.$$

Again the binomial coefficient is equal to 1, so we have

$$P(Y \geq 1) = 1 - \left(\frac{2}{3}\right)^5 = 1 - \frac{32}{243} = \frac{211}{243}.$$

5. *The probability that a Democrat will vote for Hillary Clinton in the presidential primary is* 0.5*. The probability that they will vote for Barack Obama is* 0.5*. Suppose that 40,000 Democrats vote.*

   (a) *Explain how you would use the binomial distribution to find the probability that more than 20,200 people vote for Clinton. (You do not have to make the calculations, but you should describe exactly what calculations you would make.)*

   (b) *Use the normal distribution to find an estimate for the probability from part (a). (You do not need to use the histogram correction on this question since it makes so little difference for such a large number of voters.)*

   (a) If X is the number of people voting for Clinton out of the 40000, then X is binomially distributed with $n = 40000$ and $p = 0.5$. The probability that more than 20200 vote for Clinton is therefore:

   $$P(X = 20201) + P(X = 20202) + \cdots + P(X = 39999) + P(X = 40000).$$

   To find each of these probabilities, we use the formula for the binomial distribution:

   $$P(X = k) = \binom{40000}{k} (0.5)^k (0.5)^{40000-k} = \binom{40000}{k} (0.5)^{40000}.$$

   (b) The random variable X is approximately a normal distribution with mean $\mu = np = 20,000$ and standard deviation $\sigma = \sqrt{np(1-p)} = \sqrt{10000} = 100$. The probability that X is greater than 20200 is therefore approximately

   $$P(Z \geq (20200 - 20000)/100) = P(Z \geq 2) = 1 - P(Z \leq 2) = 1 - 0.9772 = 0.0228.$$