Math 107: Calculus II, Spring 2008: Midterm Exam II Solutions

1. *(a) If* $g(x,y) = e^{2x}\cos(y)$*, find* $\dfrac{\partial^2 g}{\partial x\,\partial y}$*;*

   (b) The function $f(x,y)$ *has a critical point at* $(-1,3)$ *and Hessian matrix*

   $$Hf(-1,3) = \begin{pmatrix} -5 & 2 \\ 2 & -1 \end{pmatrix}.$$

   Is this critical point a local max, local min, or saddle point? (Explain your reasoning.)

   (c) Which of the following partial derivatives gives the slope of the tangent line at $y = 1$ *to the* $x = 2$ *cross-section of* $f(x,y)$*? (Circle your answer.)*

   $$\frac{\partial f}{\partial x}(1,2) \qquad \frac{\partial f}{\partial x}(2,1) \qquad \frac{\partial f}{\partial y}(1,2) \qquad \frac{\partial f}{\partial y}(2,1)$$

   (a) $\dfrac{\partial g}{\partial y} = -e^{2x}\sin(y)$ and so $\dfrac{\partial^2 g}{\partial x \partial y} = -2e^{2x}\sin(y)$

   (b) To classify this critical point, we find the determinant of the Hessian matrix. This is

   $$\det HF(-1,3) = (-5)(-1) - (2)(2) = 5 - 4 = 1.$$

   Since the determinant is positive, the critical point is either a local max or a local min. We now look at the top-left and bottom-right entries. These are both negative which tells us that it is a **local maximum**.

   By far the biggest mistake on this question was to assume that the eigenvalues of the matrix are -5 and -1. **This is not true!** It is not normally the case that the eigenvalues of a matrix are the top-left and bottom-right entries. (It's only true if the other two entries are both zero.) To tell this is a local max then you have to first look at the determinant. It is not enough just to say that -5 and -1 are negative.

   (c) The $x = 2$ cross-section involves keeping x constant. The slope of this cross-section is therefore related to $\dfrac{\partial f}{\partial y}$ since it tells you how f changes as y changes (and x is kept constant). Since it is the $x = 2$ cross-section, and the slope is taken at $y = 1$, this means the slope is equal to the partial derivative evaluated at $(2,1)$. So the correct answer is the last option.

2. *Let* f *be the function of two variables given by*

   $$f(x,y) = xy(x+y)$$

   (a) Calculate the gradient vector ∇f *and evaluate at the point* $(1,2)$*.*

   (b) What is the directional derivative of f *at* $(1,2)$ *in the direction of the vector* $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$*?*

(c) *Find a direction in which the directional derivative of f at $(1, 2)$ is equal to zero.*

(a) We have $f(x, y) = x^2y + xy^2$ so the partial derivatives are

$$\frac{\partial f}{\partial x} = 2xy + y^2, \qquad \frac{\partial f}{\partial y} = x^2 + 2xy.$$

Therefore the gradient vector is

$$\nabla f = \begin{pmatrix} 2xy + y^2 \\ x^2 + 2xy \end{pmatrix}$$

and evaluating at $(1, 2)$ we get

$$\nabla f(1, 2) = \begin{pmatrix} 8 \\ 5 \end{pmatrix}.$$

(b) The directional derivative is equal to

$$\frac{\begin{pmatrix} 8 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}}{\sqrt{3^2 + 1^2}} = \frac{29}{\sqrt{10}}.$$

(c) The directional derivative will be equal to zero in a direction perpendicular to the gradient vector. We therefore want to find a vector $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ such that

$$\begin{pmatrix} 8 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0.$$

This means that we want $8u_1 + 5u_2 = 0$. A possible answer is $u_1 = 5, u_2 = -8$. So the directional derivative will be zero in the direction

$$\begin{pmatrix} 5 \\ -8 \end{pmatrix}.$$

It is **not** OK to take $u_1 = 0$ and $u_2 = 0$. Even though this does give a vector that is perpendicular to the gradient vector, it does not specify a **direction**. The vector $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ does not point anywhere, so this does not answer the question of finding a direction in which the directional derivative is 0.

3. (a) *Find the linear approximation to the function $f(x, y) = \dfrac{1}{2x + y + 1}$ at $(0, 1)$.*

(b) *Use your approximation from part (a) to get an estimate of the value of $f(0.1, 0.96)$.*

(a) We want to use the formula

$$f(x,y) \simeq f(0,1) + \frac{\partial f}{\partial x}(0,1)(x-0) + \frac{\partial f}{\partial y}(0,1)(y-1).$$

We have $f(0,1) = \frac{1}{2}$. Then by the chain rule:

$$\frac{\partial f}{\partial x} = -2(2x+y+1)^{-2}, \quad \frac{\partial f}{\partial x}(0,1) = -2(2)^{-2} = -2/4 = -1/2$$

and

$$\frac{\partial f}{\partial y} = -(2x+y+1)^{-2}, \quad \frac{\partial f}{\partial y}(0,1) = -(2)^{-2} = -1/4.$$

Therefore, the linear approximation is:

$$f(x,y) \simeq \frac{1}{2} - \frac{1}{2}x - \frac{1}{4}(y-1).$$

(b) To get an estimate of $f(0.1, 0.96)$, we just substitute in for x and y:

$$f(0.1, 0.96) \simeq \frac{1}{2} - \frac{1}{2} \times 0.1 - \frac{1}{4}(0.96-1) = 0.5 - 0.05 + 0.01 = 0.46.$$

4. *Does the function $f(x,y) = e^{xy}$ have a local maximum or a local minimum? (Explain your answer as fully as possible.)*

To decide if a function has a local maximum or local minimum, we first have to find the critical points. To do this, we set the partial derivatives of f equal to zero. We have

$$\frac{\partial f}{\partial x} = ye^{xy}, \quad \frac{\partial f}{\partial y} = xe^{xy}.$$

Setting these equal to zero, we get

$$ye^{xy} = 0 \implies y = 0$$

and

$$xe^{xy} = 0 \implies x = 0.$$

This means that the only critical points of this function is at $(0,0)$.

We now have to use the second derivative test to decide what sort of critical point $(0,0)$ is. First we find the second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = y^2 e^{xy}, \quad \frac{\partial^2 f}{\partial y^2} = x^2 e^{xy}$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = xye^{xy} + e^{xy}.$$

This mixed one is tricky - you needed to use the product rule to find it. Many people put just xye^{xy} which is wrong.

Therefore the Hessian matrix is

$$\begin{pmatrix} y^2e^{xy} & xye^{xy} + e^{xy} \\ xye^{xy} + e^{xy} & x^2e^{xy} \end{pmatrix}.$$

We now evaluate the Hessian at our critical point which gives

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We apply our second derivative test to this. Again, we cannot just look at the top-left and bottom-right entries. We have to find the determinant. This is

$$(0)(0) - (1)(1) = -1$$

and since this is negative, it means that the critical point is a saddle point. It is therefore not a max or a min, and since $(0, 0)$ is the only critical point, this means that the function e^{xy} does not have a local max or local min.

This question was not done very well on the whole. Many people said that since the function e^{xy} goes off to infinity, there cannot be a max. It is correct that this tells you there is not a **global** max, but it does not mean there is not a local max. Similarly, the function e^{xy} can approach zero, but never reach it, so it does not have a global min. This does not rule out the possibility of a local min.

It is very important to learn the method of this question, because I can guarantee something similar will appear on the final. To find the local max/mins of a function, you first find the critical points, then evaluate the Hessian at each critical point and use the second derivative test.

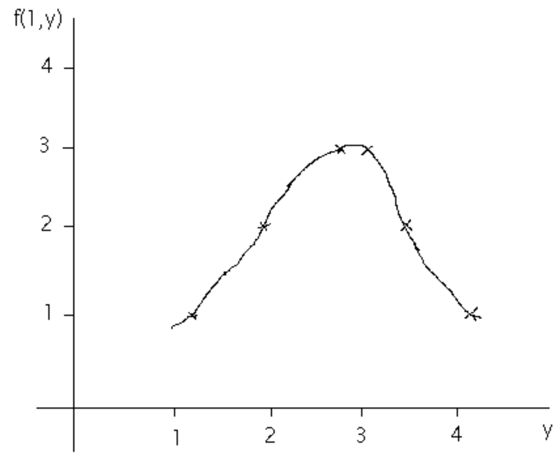
5. *The following diagram displays the c-level curves of a function $f(x, y)$ of two variables for $c = 1, 2, 3, 4$.*

   (a) *Draw a graph of the $x = 1$ cross-section of the function $f(x, y)$. (Fully label the axes of your graph and be as accurate as possible.)*

   (b) *For each of the points marked A, B, C, decide if the partial derivative $\frac{\partial f}{\partial x}$ at that point is likely to be positive, negative or zero.*

   (c) *At each of the points marked A, B, C, draw an arrow that represents the direction of the gradient vector for f at that point. (You should draw the arrows directly on the above diagram. The length of the arrows does not matter, only the direction.)*

   (a) The following graph shows what the $x = 1$ cross-section looks like. You can find this as follows. Draw the line $x = 1$ on the graph (see below) and look to see where it crosses the level curves. You can see that it crosses the level curve $c = 1$ at approximately $Y = 1$. This means that on the cross-section the graph should pass through $f(1, y) = 1$ at roughly $y = 1$. If you do the same for the other points where the line $x = 1$ crosses a level curve, you get a bunch of points which you can then join with a smooth curve.



(b) To decide if the partial derivative $\frac{\partial f}{\partial x}$ is positive, negative or zero, you look to see how the function $f(x, y)$ varies if you start at one of the points A, B, C and start moving in the positive x-direction (i.e. to the right).

Starting from A, moving to the right goes along the level curve. The function will be constant as you do this, and so the partial derivative will be equal to zero.

Starting from B, moving to the right takes you from the $c = 3$ curve into the region between the $c = 3$ and $c = 4$ curves. Therefore the function increases, and so the partial derivative will be positive.

Starting from C, moving to the right takes you from the $c = 2$ curve into the region between the $c = 1$ and $c = 2$ curves. Therefore the function decreases, and so the partial derivative is negative.

(c) The gradient vectors should be perpendicular to the level curves, and point in the direction that $f(x, y)$ is increasing, that is, towards the level curve with a *larger* value of c. See the following diagram: