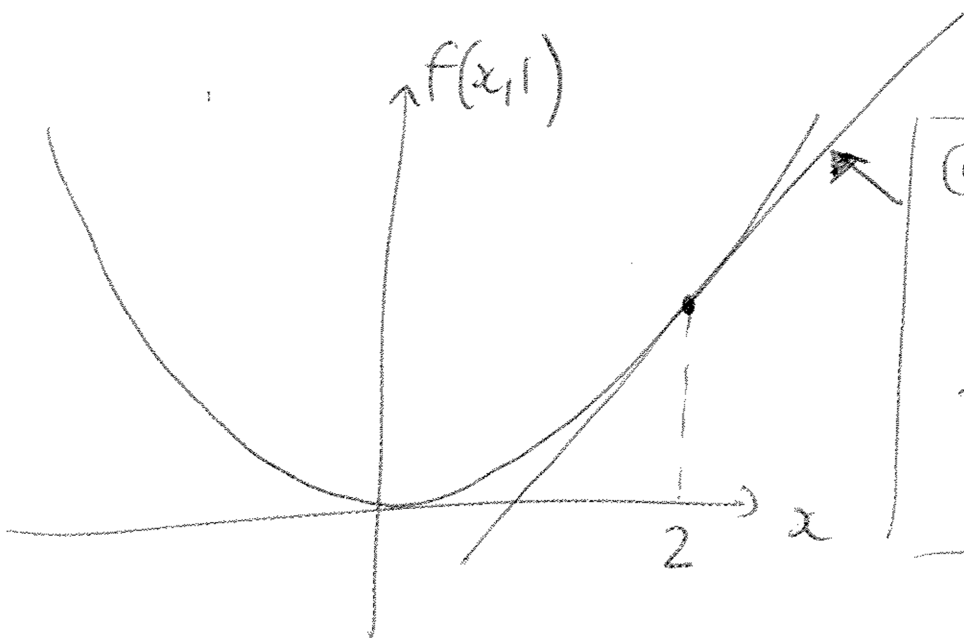
107 Midterm 2 Practice Exam 2 Solutions

(1) (a) $\frac{\partial f}{\partial x} = 2x - y + y^2$

$\frac{\partial f}{\partial y} = -x + 2xy$

(b) $f(x,1) = x^2 - x + x = x^2$



(c) $\frac{\partial f}{\partial x}(2,1) =$ slope of tangent line to cross-section at $x=2$

(d) For a saddle point, the eigenvalues must be one positive and one negative.

(2) $\frac{\partial f}{\partial x} = 3x^2 - 3$

$\frac{\partial f}{\partial y} = 2y - 2$

so critical points are where

$3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$2y - 2 = 0 \Rightarrow y = 1$

so critical points are $(1,1)$ and $(-1,1)$.

$$Hf = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}$$

so $Hf(1,1) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$. ~~Both eigen~~

det $= 12$, $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2} > 0 \Rightarrow$ local min at $(1,1)$

$Hf(-1,1) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}$, det $= -12 \Rightarrow$ saddle at $(-1,1)$

3 (a)

$$D_{\binom{1}{1}} f(4,2) = \frac{\frac{\partial f}{\partial x}(4,2) \times 1 + \frac{\partial f}{\partial y}(4,2) \times 1}{\sqrt{1^2+1^2}}$$

$$\frac{\partial f}{\partial x} = e^{x-2y} \Rightarrow \frac{\partial f}{\partial x}(4,2) = 1$$

$$\frac{\partial f}{\partial y} = -2e^{x-2y} \Rightarrow \frac{\partial f}{\partial y}(4,2) = -2$$

so $D_{\binom{1}{1}} f(4,2) = \frac{1-2}{\sqrt{2}} = \boxed{\frac{-1}{\sqrt{2}}}$

(b) In the direction of the gradient vector

$$\nabla f(4,2) = \begin{pmatrix} \frac{\partial f}{\partial x}(4,2) \\ \frac{\partial f}{\partial y}(4,2) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

(c) This is equal to the length of $\nabla f(4,2)$ which is $\sqrt{1^2+(-2)^2} = \boxed{\sqrt{5}}$

(d) Any direction perpendicular to $\begin{pmatrix}1\\-2\end{pmatrix}$.

i.e. $\begin{pmatrix}u_1\\u_2\end{pmatrix}$ such that

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\end{pmatrix} = 0$$

i.e. $\quad u_1 - 2u_2 = 0$

e.g. $u_1 = 2, \ u_2 = 1.$

So directional derivative is zero in direction $\begin{pmatrix}2\\1\end{pmatrix}$.

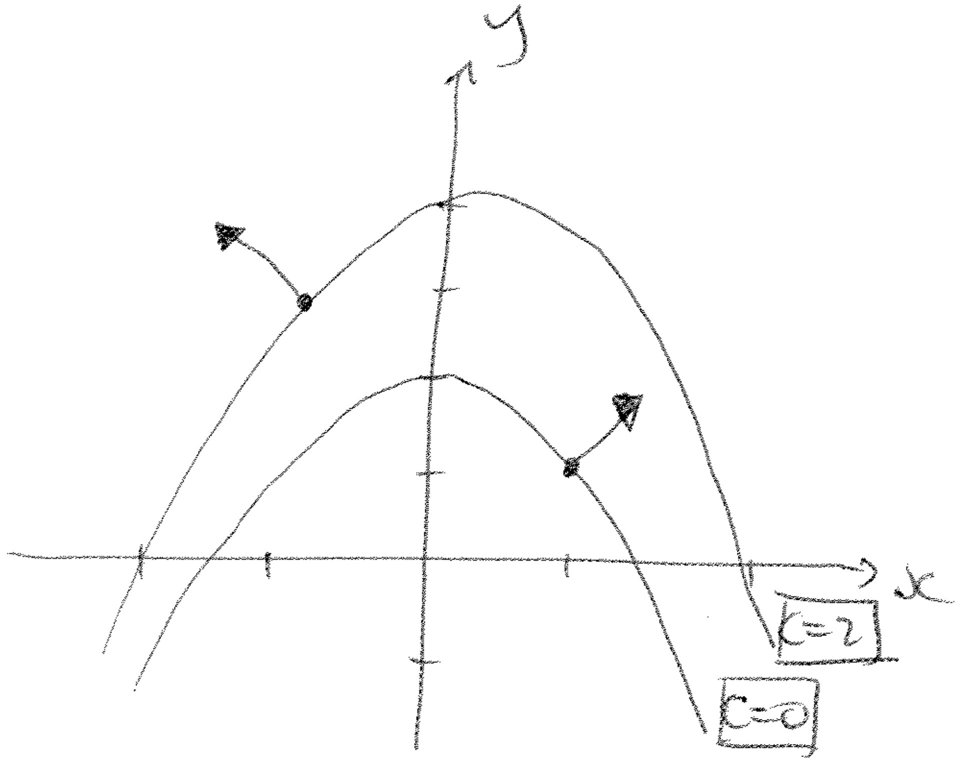
(4) c-level curve is

$$y + x^2 - 2 = c$$

or $\quad y = -x^2 + c + 2.$

$\underline{c=0}$: $y = -x^2 + 2$

$\underline{c=2}$: $y = -x^2 + 4$



(b) $\nabla f = \begin{pmatrix} 2x \\ 1 \end{pmatrix}$

(c) $f(x,y)$ does not have a global max or global min. It has no critical points since $\frac{\partial f}{\partial y} = 1$, hence not even a local max or min.

(5) We use the linear approximation at $(1,2)$.

$$f(x,y) \simeq f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$$

$$\underset{\sin(0)=0}{\parallel}$$

$$\frac{\partial f}{\partial x} = \cos(x+2y-5) \Rightarrow \frac{\partial f}{\partial x}(1,2) = \cos(0) = 1.$$

$$\frac{\partial f}{\partial y} = 2\cos(x+2y-5) \Rightarrow \frac{\partial f}{\partial y}(1,2) = 2\cos(0) = 2.$$

so $f(x,y) \simeq (x-1) + 2(y-2)$.

So $f(1.05, 2.05) \simeq 0.05 + 2 \times 0.05$

$$\simeq \boxed{0.15.}$$