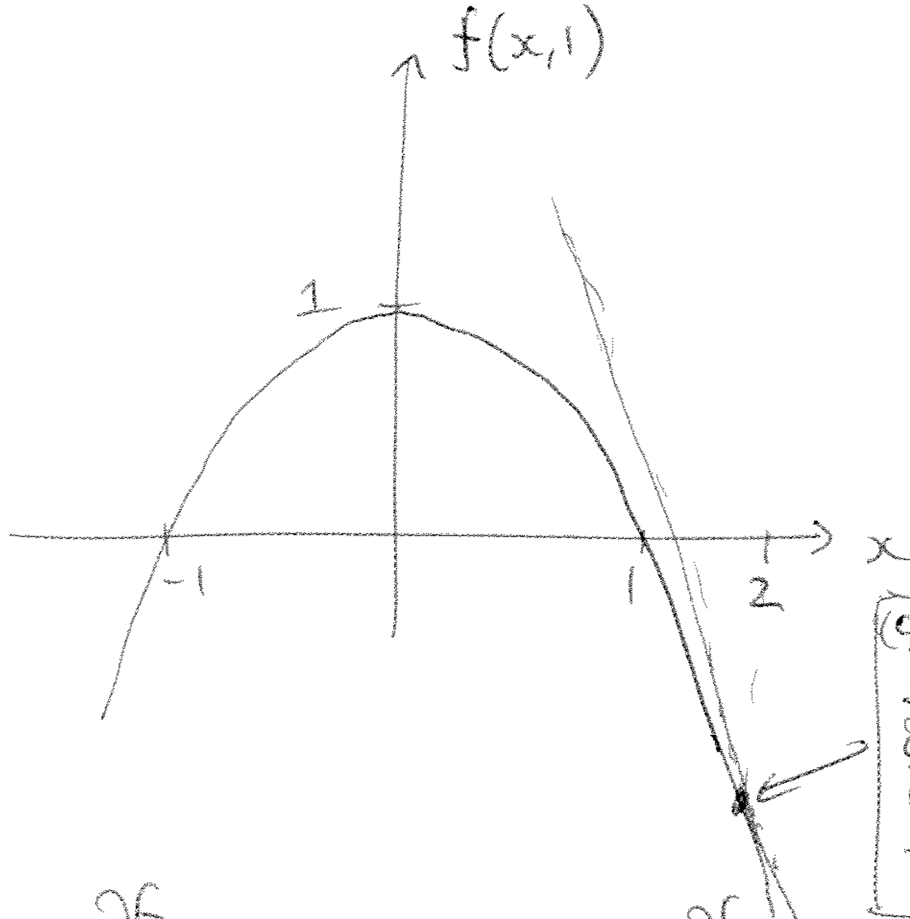
107 Midterm 2 : Practice Exam 1 Solutions

① (a) The $y=1$ cross-section is :

$$f(x,1) = 1 - x^2$$

This has graph :

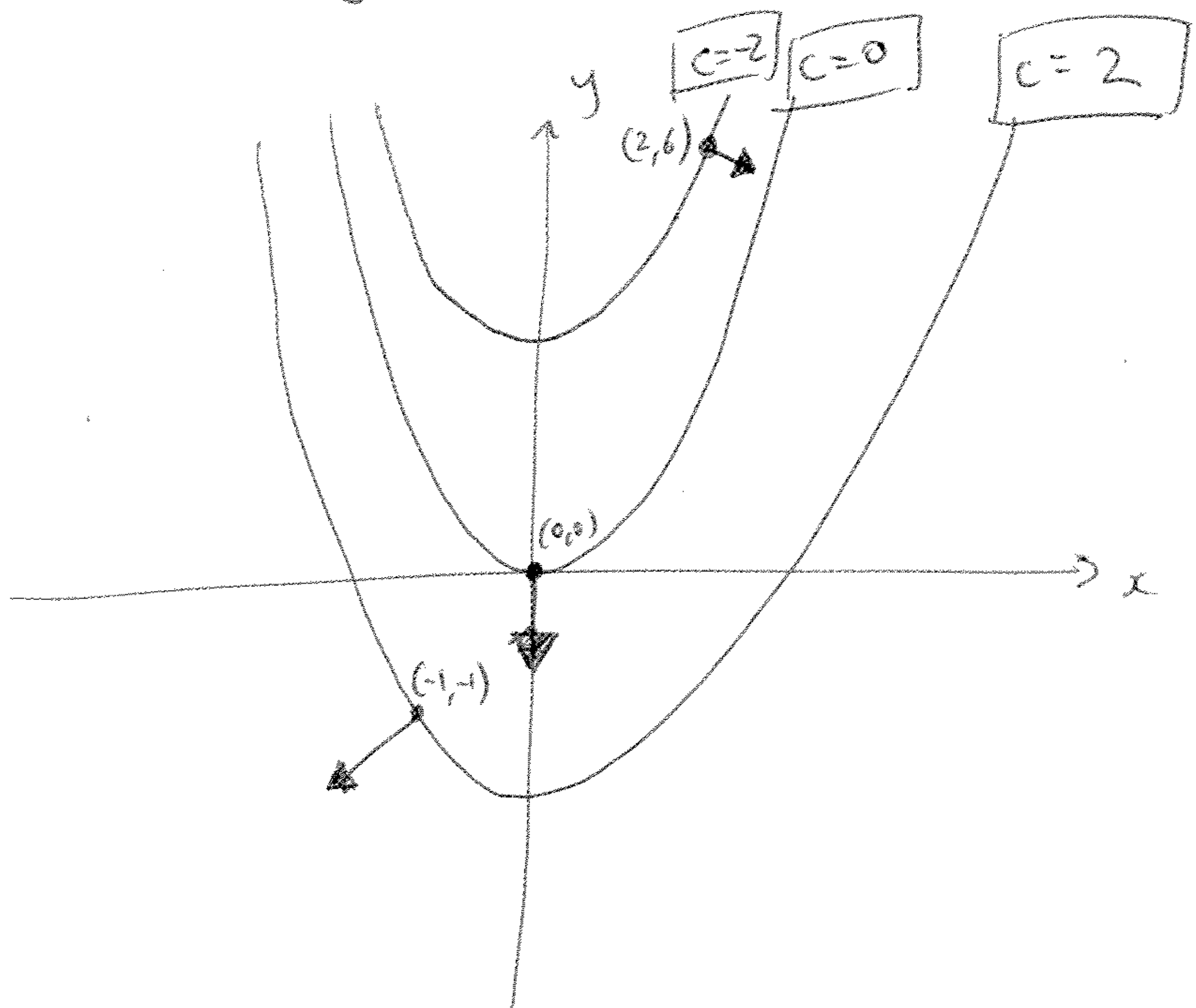


(c) $\dfrac{\partial f}{\partial x}(2,1) = -3$
is slope of
tangent line at $x=2$

(b) $\dfrac{\partial f}{\partial x} = 1 - 2xy$, $\dfrac{\partial f}{\partial x}(2,1) = 1 - 4 = -3$

② (a) c-level curve is $x^2 - y = c$

or $y = x^2 - c$



(c) The direction $\begin{pmatrix}1\\0\end{pmatrix}$ is perpendicular to the gradient vector at $(0,0)$, so directional derivative is 0.

<u>or</u> $$\frac{\frac{\partial f}{\partial x}(0,0)\,u_1 + \frac{\partial f}{\partial y}(0,0)\,u_2}{\sqrt{u_1^2+u_2^2}} = \frac{0\times 1 + -1\times 0}{1} = 0.$$

(3) $f(x,y) = x^2 + 3y^2$

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 6y$$

So the critical point is where $2x = 0$ and $6y = 0$ i.e. $(0,0)$.

<u>Hessian matrix</u>: $\frac{\partial^2 f}{\partial x^2} = 2$ $\frac{\partial^2 f}{\partial y^2} = 6$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

so Hessian is $\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$.

This has eigenvalues 2 and 6, <u>both positive</u> so $(0,0)$ is a <u>local min</u>.

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(a). $f(x,y) = e^{x^2 y}$

$$\frac{\partial f}{\partial y} = x^2 e^{x^2 y}$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = x^4 e^{x^2 y}}$$

(b) $f(x,y) = \sin(2x+y)$

$$\frac{\partial f}{\partial x} = 2\cos(2x+y)$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = -2\sin(2x+y)}$$

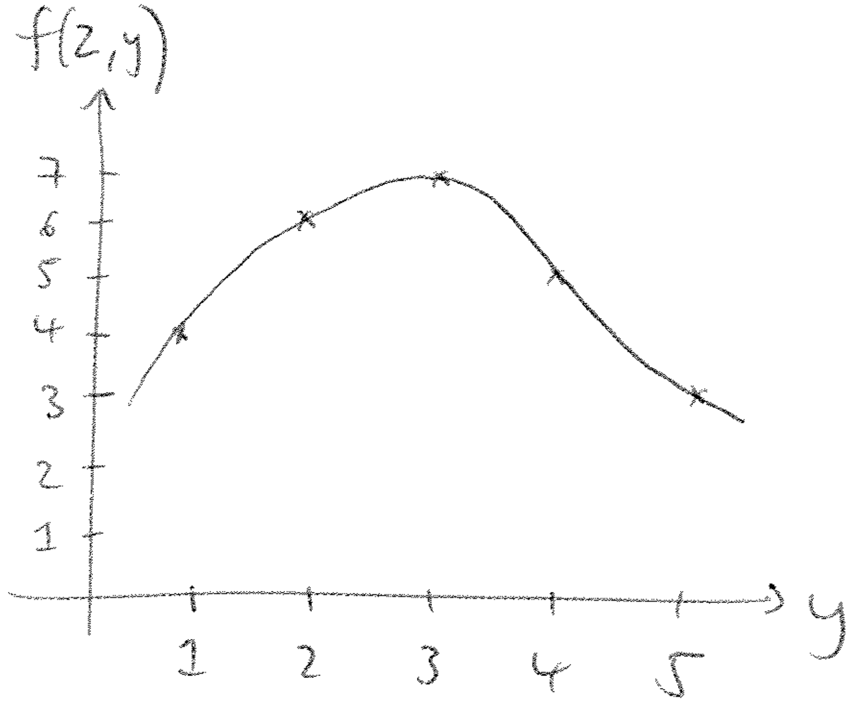
(5)

(a) (i) negative, since f decreases from 4 towards 2 in x-direction

(ii) positive, since f increases from 4 towards 5 in positive y-direction

(iii) positive, since f increases from 4 towards 7 in direction $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

(b) <u>$x=2$ cross-section</u>



<u>$y=3$ cross-section</u>

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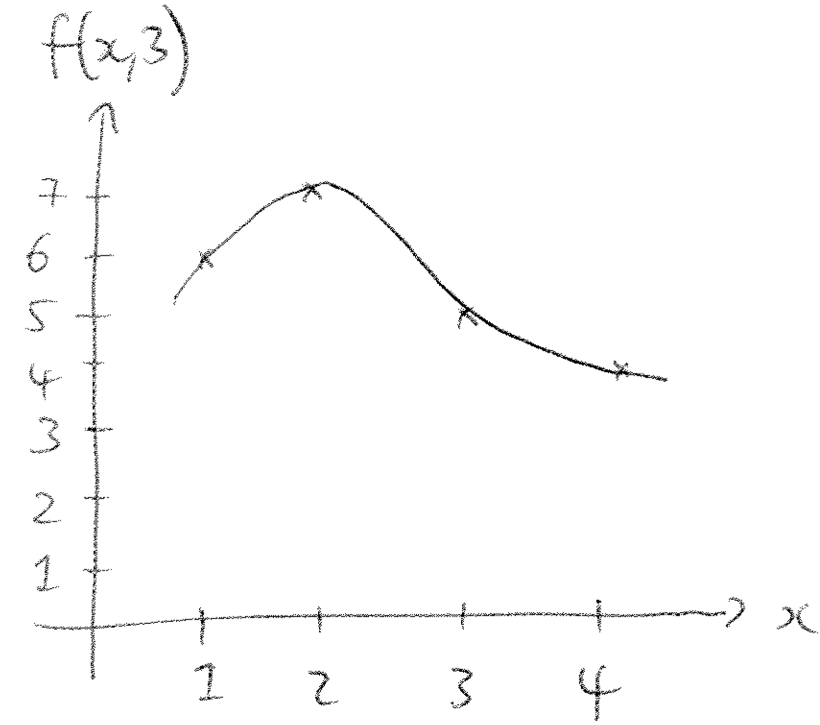

(c) $y=3$ is a max of the cross-section $f(2,y)$

$\Rightarrow \quad \frac{\partial f}{\partial y}(2,3) = 0$

$x=2$ is a max of the cross-section $f(x,3)$

$\Rightarrow \quad \frac{\partial f}{\partial x}(2,3) = 0$

so $(2,3)$ is a critical point.