1. (9 points) For the following matrices

$$A = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 \\ 0 & 8 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & -2 \\ 4 & -3 \end{bmatrix}$$

(a) Give a geometric interpretation for $T(x) = Ax$.

Rotation counterclockwise by $\pi/4$

(b) Calculate BD

$$\begin{bmatrix} 1 & -3 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} -13 & 7 \\ 32 & -24 \end{bmatrix}$$

(c) Find B^{-1}

$$\begin{bmatrix} 1 & -3 \\ 0 & 8 \end{bmatrix}^{-1} = \frac{1}{8}\begin{bmatrix} 8 & 3 \\ 0 & 1 \end{bmatrix}$$

2. (6 points) Solve the following system of equations

$$\begin{array}{llll} x & +y & +z & = 1 \\ 3x & -y & +3z & = -1 \\ 2x & & -z & = 1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & 3 & -1 \\ 2 & 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_3-2R_2]{R_2-3R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 0 & -4 \\ 0 & -2 & -3 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2/-4}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & -3 & -1 \end{bmatrix} \xrightarrow[R_3+2R_2]{R_1-R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3/-3}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/3 \end{bmatrix} \xrightarrow{R_1-R_3} \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/3 \end{bmatrix}$$

$$(x,y,z) = \left(\tfrac{1}{3}, 1, -\tfrac{1}{3}\right)$$

3. (9 points) Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 3 \\ -1 & 6-\lambda \end{pmatrix}$$

$$= (2-\lambda)(6-\lambda) + 3$$

$$= 12 - 8\lambda + \lambda^2 + 3$$

$$= \lambda^2 - 8\lambda + 15$$

$$= (\lambda - 3)(\lambda - 5)$$

$\lambda = 3, 5$

$\lambda = 3$

$$A - \lambda I = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}$$

$-x + 3y = 0$

$x = 3y$

$$\boxed{\lambda = 3 \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

$\lambda = 5$

$$A - \lambda I = \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix}$$

$-3x + 3y = 0$

$\Rightarrow x = y$

$$\boxed{\lambda = 5 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

4. (6 points) Solve the the following system of differential equations

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = -x + 6y$$

with the initial conditions $x(0) = 5$, $y(0) = 3$

Using problem 3's answer

$$\begin{bmatrix} x \\ y \end{bmatrix}(t) = c_1 e^{3t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$3c_1 + c_2 = 5$$
$$c_1 + c_2 = 3$$

$$\begin{bmatrix} 3 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 0 & -2 & -4 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow -2c_2 = -4$$
$$c_2 = 2$$

$$c_1 + 2 = 3$$
$$\Rightarrow c_1 = 1$$

$$\boxed{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{3t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

5. (6 points) For the matrix $A = \begin{bmatrix} -2 & 3 \\ 0 & -4 \end{bmatrix}$

(a) Determine the stability of the system $\vec{x}'(t) = A\vec{x}(t)$,

$$\det(A - \lambda I) = \det\begin{bmatrix} -2-\lambda & 3 \\ 0 & -4-\lambda \end{bmatrix}$$

$$= (-2-\lambda)(-4-\lambda) + 0$$

$$= 8 + 6\lambda - \lambda^2 = (\lambda+2)(\lambda+4)$$

$$\Rightarrow \lambda = -2, -4$$

~~do those~~ . both are < 0

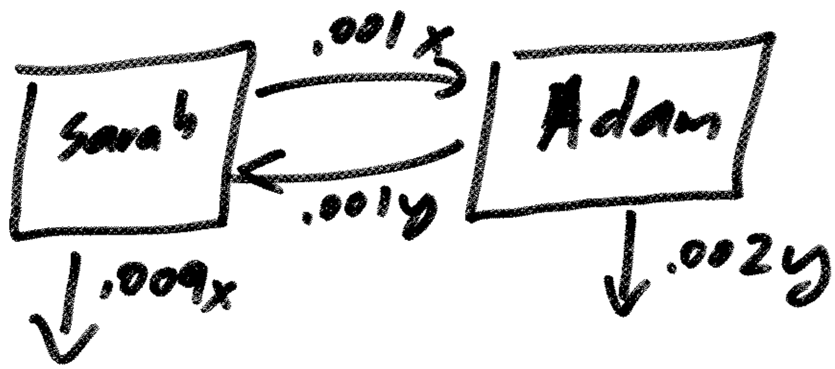
$\Rightarrow$ System is stable

(b) If $\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, find $\lim_{t\to\infty} \vec{x}(t)$ (Hint: you do not need to solve the system)

Since the system is stable

$$\lim_{t \to \infty} \vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

6. (6 points) Sarah and Adam own competing timium refineries located on the Brianese river. At any given time t, Sarah and Adam have $x(t)$ kg and $y(t)$ kg of pollutant in their respective plants. Pollutants from Sarah's factory drain into the river at a rate of $.009x$ kg/min and pollutants from Adam's drains at a rate of $.002y$ kg/min. Also, due to proximity and shady business practices, pollutants from Sarah's factory flow into Adam's at a rate of $.001x$ kg/min and from Adam's factory into Sarah's at a rate of $.001y$ kg/min. Assuming no more pollutants are being added, set up a system of differential equations describing this model.



$$\frac{dx}{dt} = r_{in}^{x} - r_{out}^{x}$$

$$r_{in}^{x} = .001y$$

$$r_{out}^{x} = .001x + .009x$$

$$= .01x$$

$$\frac{dy}{dt} = r_{in}^{y} - r_{out}^{y}$$

$$r_{in}^{y} = .001x$$

$$r_{out} = .001y + .002y$$

$$= .003y$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= .001y - .01x \\ \frac{dy}{dt} &= .001x - .003y \end{aligned}}$$

7. (6 points) Let A be a 2x2 matrix such that

$$A\begin{bmatrix}1\\2\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix} \qquad\qquad A\begin{bmatrix}4\\-1\end{bmatrix} = 1/7\begin{bmatrix}4\\-1\end{bmatrix}$$

(a) Show that A is invertible (Hint: You do not need to know exactly what A is)

A has eigenvalues $3, \frac{1}{7}$

$\Rightarrow \det(A) = 3 \cdot \frac{1}{7} \neq 0$

$\Rightarrow$ A is invertible

(b) Find the eigenvectors and eigenvalues of A^{-1}

$$A^{-1}\left(A\begin{bmatrix}1\\2\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix}\right)$$

$$\begin{bmatrix}1\\2\end{bmatrix} = A^{-1}\left(3\begin{bmatrix}1\\2\end{bmatrix}\right)$$

$$= 3A^{-1}\begin{bmatrix}1\\2\end{bmatrix}$$

$$\Rightarrow A^{-1}\begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\2\end{bmatrix}$$

$\Rightarrow \frac{1}{3}$ is an eigenvalue w/eigenvector $\begin{bmatrix}1\\2\end{bmatrix}$

likewise

7 is an eigenvalue w/eigenvector $\begin{bmatrix}4\\-1\end{bmatrix}$

1. (4 points) Find the following indefinite integral

$$\int (x^2+1)e^{3x}dx$$

$u = x^2+1 \quad v = \frac{1}{3}e^{3x}$

$du = 2x\,dx \quad dv = e^{3x}dx$

$\int (x^2+1)e^{3x}dx = (x^2+1)(\frac{1}{3}e^{3x}) - \int \frac{1}{3}e^{3x}\,2x\,dx$

$u = 2x \quad v = \frac{1}{9}e^{3x}$

$du = 2dx \quad dv = \frac{1}{3}e^{3x}dx$

$= \frac{1}{3}(x^2+1)e^{3x} - \left[\frac{2}{9}xe^{3x} - \frac{2}{9}\int e^{3x}dx\right]$

$= \boxed{\frac{1}{3}(x^2+1)e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C}$

2. (6 points) Find if the following integrals converge or do not converge

(a) $\int_0^6 \frac{1}{(x-3)^2}dx$

$$= \underbrace{\lim_{t\to 3^-} \int_0^t \frac{1}{(x-3)^2}dx}_{} + \lim_{t\to 3^+} \int_t^6 \frac{1}{(x-3)^2}dx$$

$$\lim_{t\to 3^-} \left[\frac{-1}{x-3}\right]_0^t = \lim_{t\to 3^-} \frac{-1}{t-3} + \frac{1}{-3} = \infty$$

$$\Rightarrow \boxed{\text{does not converge}}$$

(b) $\int_0^\infty xe^{-3x}dx$ (you can use the fact that $lim_{x\to\infty}xe^{-x} = 0$)

$u = x \quad v = -\frac{1}{3}e^{-3x}$

$du = dx \quad dv = e^{-3x}dx$

$$\lim_{t\to\infty} \int_0^t xe^{-3x}dx = \lim_{t\to\infty} -\frac{1}{3}xe^{-3x}\Big|_0^t + \frac{1}{3}\int_0^t e^{-3x}dx$$

$$= \lim_{t\to\infty} \underbrace{-\frac{1}{3}te^{-3t}}_{=0} - \frac{1}{9}e^{-3x}\Big|_0^t$$

$$= \lim_{t\to\infty} \underbrace{-\frac{1}{9}e^{-3t}}_{=0} + \frac{1}{9} = \boxed{\frac{1}{9}}$$

(c) $\int_{-\infty}^9 \frac{x}{x^2+3}dx$

$$= \lim_{t\to-\infty} \int_t^9 \frac{x}{x^2+3} = \lim_{t\to-\infty}\frac{1}{2}\int \frac{du}{u} = \lim_{t\to-\infty} \ln|u|$$

$u = x^2+3$

$du = 2dx$

$$= \lim_{t\to-\infty} \ln(x^2+3)\Big|_t^9$$

$$= \lim_{t\to-\infty} \underbrace{\ln(t^2+3)}_{=\infty} + \ln 30$$

$$= \infty \Rightarrow \boxed{dnc}$$

3. (5 points) Find the equilibria for the following differential equation and state weather each is stable or unstable.

$$\frac{dy}{dt} = (y - y^2)\ln\left(y^4 + \frac{15}{16}\right)$$

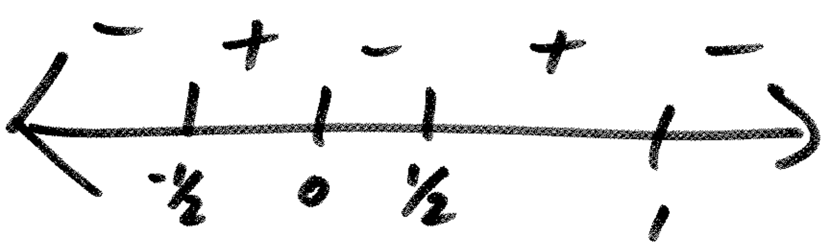
$$= y(1-y)\ln\left(y^4 + \frac{15}{16}\right)$$

$y = 0$ or $y^4 + \frac{15}{16} = 1$

$y = 1$

$y^4 = \frac{1}{16}$

$\Rightarrow y = \pm\frac{1}{2}$



$y = 1$ stable

$y = \frac{1}{2}$ unstable

$y = 0$ stable

$y = -\frac{1}{2}$ unstable

4. (5 points) Find the third degree taylor polynomial for $(x+1)^{1/3}$ centered at 0 and use it to approximate $(1.1)^{1/3}$ (you do not need to simplify this approximation).

$f(x) = (x+1)^{1/3}$ $\quad$ $f(0) = 1$

$f'(x) = \frac{1}{3}(x+1)^{-2/3}$ $\quad$ $f'(0) = \frac{1}{3}$

$f''(x) = \frac{-2}{9}(x+1)^{-5/3}$ $\quad$ $f''(0) = \frac{-2}{9}$

$f'''(x) = \frac{10}{27}(x+1)^{-8/3}$ $\quad$ $f'''(0) = \frac{10}{27}$

$\Rightarrow \quad T_3(x) = 1 + \frac{1}{3}x + \frac{-2}{9}x^2 + \frac{10}{27}x^3$

$(1.1)^{1/3} \approx T(.1) = 1 + \frac{1}{3}(.1) + \frac{-2}{9}(.1)^2 + \frac{10}{27}(.1)^3$

5. (5 points) Solve the following differential equation.

$$\frac{dy}{dt} = 2t(y + 3y^2) \;\; ; \;\; y(0) = 1$$

$$\int \frac{dy}{y(1+3y)} = \int 2t\,dt$$

$$\frac{A}{y} + \cancel{1+3}\ \frac{B}{1+3y} = \frac{1}{y(1+3y)}$$

$\cancel{A+3Ay+}$

$$A(1+3y) + By = 1$$

$y = 0 \quad A = 1$

$y = -\frac{1}{3} \quad -\frac{1}{3}B = 1$

$\Rightarrow B = -3$

$$\int \frac{1}{y} + \frac{-3}{1+3y}\,dy = t^2 + c$$

$$\ln y + -\ln(1+3y) = t^2 + c$$

$$\ln\left(\frac{y}{1+3y}\right) = t^2 + c$$

$$\frac{y}{1+3y} = Ce^{t^2}$$

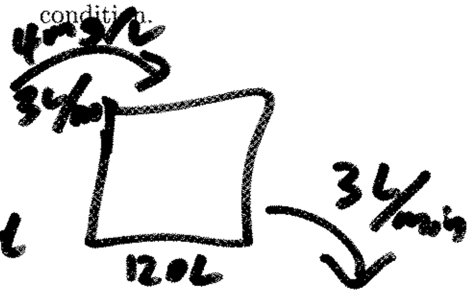
$y(0) = 1$

$\frac{1}{4} = \frac{1}{1+3\cdot 1} = C$

$$\frac{y}{1+3y} = \frac{1}{4}e^{t^2}$$

6. (6 points) Tim has a 120 L fish tank which initially contains a concentration of 5mg/L of ammonia. In order to purify this, Tim removes water from the tank at 3 L/minute and replaces it with filtered water containing only .04mg/L of ammonia at the same rate.

(a) (2 points) Set up a differential equation appropriate to this system. Include the initial value condition.



X = Amt

$\frac{x}{120}$ = concentration

$r_{in} = 4\,\text{mg/L} \cdot 3\,\text{L/min} = 12\,\text{mg/min}$

$r_{out} = 3\,\text{L/min} \cdot \frac{x}{120}\,\text{mg/L} = \frac{x}{40}\,\text{mg/min}$

(b) (4 points) Use this system to find the **concentration** of ammonia in the tank as a function of time

$$\boxed{\frac{dx}{dt} = 12 - \frac{x}{40} \qquad x(0) = 120 \cdot 5 = \cancel{120}\ 600}$$

$$\int \frac{dx}{12 - \frac{x}{40}} = \int dt$$

$$-40 \ln\left(12 - \frac{x}{40}\right) = t + C$$

$$\ln\left(12 - \frac{x}{40}\right) = \frac{-t}{40} + C$$

$$12 - \frac{x}{40} = Ce^{-t/40}$$

$$-\frac{x}{40} = -12 + Ce^{-t/40}$$

$$x = 480 + Ce^{-t/40}$$

$$X(0) = 600 = 480 + C \Rightarrow C = 120$$

$$X(t) = 480 + 120e^{-t/40}$$

$$\cancel{C(t) = \frac{x}{120} = 4 + e^{-t/40}}$$

$$\boxed{C(t) = 4 + e^{-t/40}}$$