Math 107: Calculus II, Spring 2008: Midterm Exam I Solutions

1. *This question is about the autonomous differential equation*

$$\frac{dy}{dt} = y(y^2+1).$$

(a) Find the equilibrium solution to this differential equation.

(b) Use the first derivative test to decide if the equilibrium is stable or unstable.

(c) Consider the solution to this equation that has $y=-1$ at $t=0$. Is this solution increasing, decreasing or constant? Explain your answer.

(a) The equilibrium solutions occur where $\frac{dy}{dt}=0$. This means we have $y(y^2+1)=0$. Now y^2+1 is never equal to zero (for real numbers y) so the only possibility is $y=0$. Therefore $\boxed{y=0}$ is the only equilibrium.

(b) The first derivative test says that we should take the derivative of the function $g(y) = y(y^2+1)$ and evaluate it at the equilibrium. Multiplying out, we get $g(y) = y^3 + y$ and so its derivative is $g'(y) = 3y^2+1$. At the equilibrium $y=0$, we have

$$g'(0) = 1.$$

This is positive which tells us that the equilibrium is $\boxed{\text{unstable}}$.

(c) At the point where $y=-1$, we have

$$\frac{dy}{dt} = -1((-1)^2+1) = -2.$$

Since the derivative is negative, this tells us that the solution is $\boxed{\text{decreasing}}$.

The main mistake made on this question was to plug $y=-1$ into $g'(y) = 3y^2+1$. But to see if a function $y(t)$ is increasing or decreasing, you have to look to see if $\frac{dy}{dt}$ is positive or negative. In this question, $\frac{dy}{dt} = y(y^2+1)$ so that is where you should substitute $y=-1$.

2. *Find the general solution to the following separable differential equation:*

$$\frac{dy}{dt} = 2ty^2.$$

To solve equations like this, we separate the variables: y on the left, and t on the right. This gives us:

$$\frac{1}{y^2}\frac{dy}{dt} = 2t.$$

We then integrate both sides with respect to the t. On the left-hand side, that results in the integral of $1/y^2$ with respect to y:

$$\int \frac{1}{y^2}\,dy = \int 2t\,dt.$$

One of the main mistakes made on this question was the failure to integrate $1/y^2$ correctly. It is **not** $\ln|y^2|$!!!!!!! It is not true that when you integrate "1 over something", you get an ln. This only happens when you integrate $1/y$. Instead, you should think of $1/y^2$ as y^{-2} and then integrate using the usual rule for powers: $y^{n+1}/(n+1)$. With $n = -2$, this gives:

$$\int \frac{1}{y^2}\, dy = \frac{y^{-1}}{-1} = \frac{-1}{y}.$$

Altogether then, we get:

$$\frac{-1}{y} = t^2 + c.$$

We now rearrange to get $y(t)$:

$$y = \frac{-1}{t^2 + c}.$$

Finally, the other big mistake people made was to forget to look for constant solutions. These are the solutions that were 'lost' when we divided by y^2, because we might have been dividing by zero. These occur then when $y^2 = 0$, or $y = 0$. Therefore, the full general solution is

$$\boxed{y = \frac{-1}{t^2 + c} \quad \text{or} \quad y = 0}.$$

3. *(a) Find the inverse of the matrix* $\begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix}$.

   (b) Find the solution to the following matrix equation

$$\begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

   (c) Suppose that the matrix equation $A\mathbf{x} = \mathbf{b}$ has no solutions. What does this tell you about the matrix A?

(Note: the matrix A in part (c) is different from the matrix appearing in the previous parts of this question.)

(a) The first thing we should check is that the inverse exists. We do this by finding the determinant of the matrix. This is

$$(-4)(1) - (-2)(3) = -4 + 6 = 2.$$

Because this is not zero, this matrix does have an inverse. We can then find it using the formula:

$$\begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix}^{-1} = \boxed{\frac{1}{2}\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} 1/2 & 1 \\ -3/2 & -2 \end{pmatrix}}.$$

(b) Whenever we have a matrix equation of the form $A\mathbf{x} = \mathbf{b}$ where A has an inverse, we can solve it by multiplying by that inverse. This gives $\mathbf{x} = A^{-1}\mathbf{b}$. In this case we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 1 \\ -3/2 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ -5 \end{pmatrix}}.$$

(c) This tells you that

$$\boxed{\text{the matrix } A \text{ is singular, i.e. the determinant of } A \text{ is zero}}.$$

4. *(a) Show that* $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ *is an eigenvector of the matrix* $\begin{pmatrix} -5 & -12 \\ 2 & 5 \end{pmatrix}$ *and find the corresponding eigenvalue.*

(b) Another eigenvector for the matrix $\begin{pmatrix} -5 & -12 \\ 2 & 5 \end{pmatrix}$ *is given by* $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ *with corresponding eigenvalue 1. Use this information, together with your answer to part (a), to write down the general solution to the following system of differential equations.*

$$\frac{dx}{dt} = -5x - 12y; \quad \frac{dy}{dt} = 2x + 5y.$$

(c) Find the specific solution to this system that satisfies $x(0) = -1, y(0) = 1$.

(a) To show that something is an eigenvector, we have to multiply it by the matrix and show we get a constant times what we started with. When we do that here, we get:

$$\begin{pmatrix} -5 & -12 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

This means that

$$\boxed{\begin{pmatrix} 3 \\ -1 \end{pmatrix} \text{ is an eigenvector with eigenvalue } -1}.$$

(b) The general solution to this system is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{k_1 t} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + c_2 e^{k_2 t} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

where k_1, k_2 are the eigenvalues and $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ the corresponding eigenvectors. Plugging in the information from the question together with what we found in part (a), we get

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}}.$$

We can write this matrix equation as separate equations for x and y:

$$x(t) = 3c_1 e^{-t} + 2c_2 e^{t}, \quad y(t) = -c_1 e^{-t} - c_2 e^{t}.$$

(c) To find the specific solution, we substitute in the values $t = 0$, $x = -1$, $y = 1$. This gives:

$$-1 = 3c_1 + 2c_2, \quad 1 = -c_1 - c_2.$$

You can solve these any way you like. Multiplying the second equation by 3 and adding them together we get:

$$2 = -c_2$$

so that $c_2 = -2$. Then $c_1 = 1$. Putting these values of c_1 and c_2 back into the equation for x and y, we get:

$$\boxed{x(t) = 3e^{-t} - 4e^{t}, \quad y(t) = -e^{-t} + 2e^{t}}.$$

5. *I want to invest \$100 that I recently won in the lottery. My three main options pay interest on my investment via the following differential equations (where $M(t)$ is the amount of my investment (measured in dollars) at time t):*

- *Bank A:* $\dfrac{dM}{dt} = (50 - M)$
- *Bank B:* $\dfrac{dM}{dt} = (M - 50)$
- *Bank C:* $\dfrac{dM}{dt} = (M - 200)$

(a) *Sketch the solutions to each of the above differential equations.*

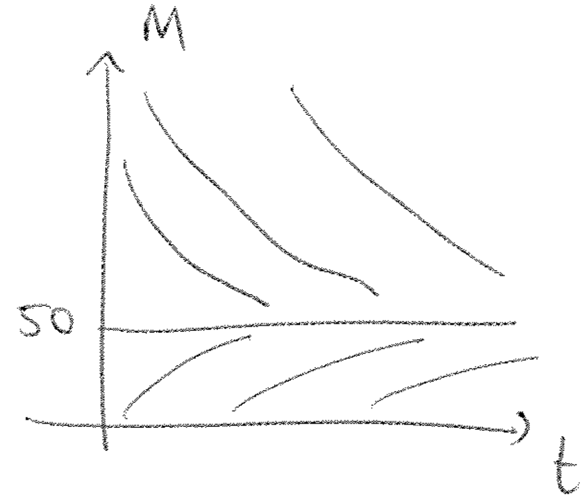
(b) *Which bank should I invest in to maximize the return from my \$100? Explain your answer.*

(a) See next page for the sketches of solutions.

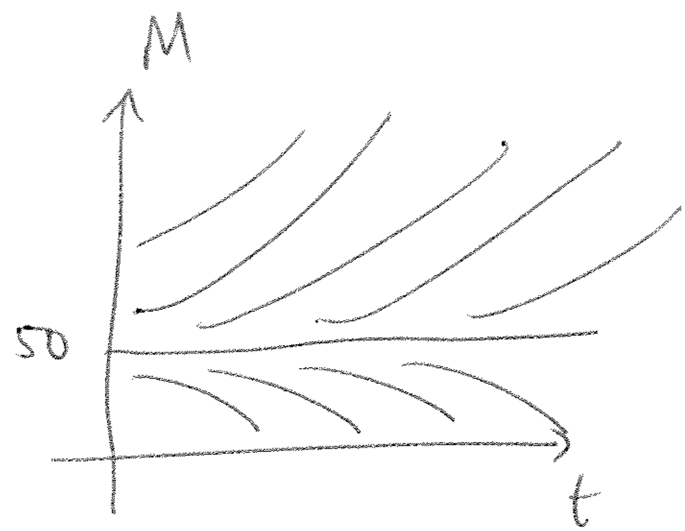
(b) In Bank A, if I start with \$100, it will decrease over time and approach \$50. In Bank C, my \$100 will decrease to zero. In Bank B, my money will increase over time. Therefore, $\boxed{\text{Bank B is the best option}}$.

Midterm I: Question 5(a)

Bank A



Bank B



Bank C