TEST 2 (04/19/2013, MATH 107, CALCULUS II (BIO))

Name: **Section:** **Score:**

In agreeing to take this exam, you are implicitly agreeing to act with fairness and honesty.

Problems/Points	1/10	2/20	3/10	4/20	5/20	6/20
Scores						

1. (10 points) Determine whether the following statements are true (T) or false (F) (You need not give an explanation). (**Each one is of 2 points**)

(a) If the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of a function $f(x, y)$ exist at a point (x_0, y_0), this function must be differentiable at (x_0, y_0).
(b) If $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).
(c) Suppose that $f(x, y)$ is a differentiable function. The gradient vector of f at a point (x_0, y_0) is not perpendicular to the level curve through (x_0, y_0).
(d) If (x_0, y_0) is a critical point of $f(x, y)$, then f must be differentiable at (x_0, y_0).
(e) If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are continuous on an open disk centered at the point (x_0, y_0), then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

Problems	(a)	(b)	(c)	(d)	(e)
T/F	F	T	F	F	T

2. (20 points) Consider the function

$$f(x,y) = \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{x^2}{2y}\right).$$

(a) **(2 points + 2 points)** Find the largest possible domain and corresponding range of f.

(b) **(4 points + 4 points)** Compute $f_x(x,y)$ and $f_y(x,y)$.

(c) **(8 points)** Verify $f_y(x,y) = \frac{1}{2} f_{xx}(x,y)$.

Proof. (a) By definition, we must have $y > 0$. Hence the largest possible domain is

$$D = \{(x,y) \in \mathbf{R}^2 : x \in \mathbf{R} \text{ and } y > 0\}.$$

Since $-x^2/2y \leq 0$, it follows that $\exp(-x^2/2y) \in (-\infty, 1)$ and then the range of f is $(0, \infty)$.

(b) Compute

$$\begin{aligned}
f_x(x,y) &= \frac{1}{\sqrt{2\pi y}} \frac{\partial}{\partial x} e^{-\frac{x^2}{2y}} &&= \frac{1}{\sqrt{2\pi y}} \cdot e^{-\frac{x^2}{2y}} \cdot \frac{-2x}{2y} \\
&= \frac{1}{\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} \frac{-x}{y} &&= -\frac{x}{y\sqrt{2\pi y}} e^{-\frac{x^2}{2y}}, \\
f_y(x,y) &= \frac{\partial}{\partial t}\left((2\pi y)^{-1/2}\right) e^{-\frac{x^2}{2y}} + \frac{1}{\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} \frac{x^2}{2y^2} \\
&= \frac{-1}{2}(2\pi y)^{-3/2} 2\pi e^{-\frac{x^2}{2y}} + \frac{x^2}{2y^2\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} \\
&= \frac{1}{2\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} \left(-\frac{1}{y} + \frac{x^2}{y^2}\right).
\end{aligned}$$

(c) From part (b), we have

$$\begin{aligned}
f_{xx}(x,y) &= \frac{\partial}{\partial x}\left(-\frac{x}{y\sqrt{2\pi y}} e^{-\frac{x^2}{2y}}\right) \\
&= \frac{-1}{y\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} + \frac{-x}{y\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} \frac{-2x}{y} \\
&= \frac{-1}{y\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} + \frac{x^2}{y^2\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} \\
&= \frac{1}{\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} \left(-\frac{1}{y} + \frac{x^2}{y^2}\right).
\end{aligned}$$

Therefore $f_y(x,y) = \frac{1}{2} f_{xx}(x,y)$. □

3. (10 points) Prove that

$$\lim_{(x,y)\to(0,0)} \frac{xy^4 + 3x^2y^2}{x^3 + 2y^6}$$

does not exist.

Proof. We first consider a line $y = mx$ with $m \neq 0$. Then

$$\begin{aligned}
\lim_{(x,y)\to(0,0),\text{ along with } y=mx} \frac{xy^4 + 3x^2y^2}{x^3 + 2y^6} &= \lim_{x\to 0} \frac{m^4x^5 + 3m^2x^4}{x^3 + 2m^6x^6} \\
&= \lim_{x\to 0} \frac{m^4x^2 + 3m^2x}{1 + 2m^6x^3} \\
&= \frac{0+0}{1+0} = 0.
\end{aligned}$$

Now we consider a parabola $x = y^2$. Then

$$\begin{aligned}
\lim_{(x,y)\to(0,0),\text{ along with } x=y^2} \frac{xy^4 + 3x^2y^2}{x^3 + 2y^6} &= \lim_{y\to 0} \frac{y^6 + 3y^6}{y^6 + 2y^6} \\
&= \lim_{y\to 0} \frac{4y^6}{3y^6} = \frac{4}{3}.
\end{aligned}$$

Thus the limit does not exist. □

4. (20 points) Consider the function

$$f(x,y) = 3xy - x^3 - y^3.$$

(a) **(10 points)** Find all critical points of $f(x,y)$.
(b) **(10 points)** Determine the type of each critical point.

Proof. (a) Since $f(x,y)$ is differentiable at any points of $\mathbf{R}^2$, it follows that all critical points of $f(x,y)$ must satisfy

$$\nabla f(x_0,y_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From

$$f_x(x,y) = 3y - 3x^2, \quad f_y(x,y) = 3x - 3y^2,$$

we get

$$y_0 - x_0^2 = 0, \quad x_0 - y_0^2 = 0$$

for any critical point (x_0,y_0) of f. Hence $(x_0,y_0) = (0,0)$ or $(1,1)$.

(b) The Hessian matrix of f at (x_0,y_0) is given by

$$H = \mathbf{Hess}(f)(x_0,y_0) = \begin{bmatrix} -6x & 3 \\ 3 & -6y \end{bmatrix}.$$

If $(x_0,y_0) = (0,0)$, then $\det H = 0 - 9 = -9 < 0$; so $(0,0)$ is a saddle point of f.

If $(x_0,y_0) = (1,1)$, then $\det H = 36 - 9 = 27 > 0$ and $f_{xx}(1,1) = -6$; so $f(x,y)$ has a local maximum at $(1,1)$. $\square$

5. (20 points) Consider the function

$$f(x,y) = y(\cos x)^2 + xe^{x^2+y}.$$

(a) **(10 points)** Compute the directional derivative of $f(x,y)$ at $(0,1)$ in the direction $\begin{bmatrix}3\\4\end{bmatrix}$.

(b) **(10 points)** Find a unit vector that is perpendicular to the level curve of $f(x,y)$ at the point $(0,0)$.

Proof. The gradient of f is

$$\begin{aligned}\nabla f(x,y) &= \begin{bmatrix}2y\cos x(-\sin x) + e^{x^2+y} + xe^{x^2+y}\cdot 2x\\ (\cos x)^2 + xe^{x^2+y}\end{bmatrix}\\ &= \begin{bmatrix}-2y\cos x\sin x + (1+2x^2)e^{x^2+y}\\ (\cos x)^2 + xe^{x^2+y}\end{bmatrix}.\end{aligned}$$

(a) At the point $(0,1)$ we have

$$\nabla f(0,1) = \begin{bmatrix}0+e^{0+1}\\ 1+0\end{bmatrix} = \begin{bmatrix}e\\1\end{bmatrix}.$$

the vector $\begin{bmatrix}3\\5\end{bmatrix}$ is not unit, we normalize it by

$$\mathbf{u} = \frac{1}{\left|\begin{bmatrix}3\\4\end{bmatrix}\right|}\begin{bmatrix}3\\4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}3/5\\4/5\end{bmatrix}.$$

The directional derivative of $f(x,y)$ at $(0,1)$ in the direction of $\begin{bmatrix}3\\4\end{bmatrix}$ is

$$D_{\mathbf{u}}f(0,1) = \nabla f(0,1)\cdot\mathbf{u} = \begin{bmatrix}e\\1\end{bmatrix}\cdot\begin{bmatrix}3/5\\4/5\end{bmatrix} = \frac{3e+4}{5}.$$

(b) The gradient of f at $(0,0)$ is perpendicular to the level curve of $f(x,y)$ at $(0,0)$, so such a unit vector is

$$\frac{1}{|\nabla f(0,0)|}\nabla f(0,0) = \frac{1}{\left|\begin{bmatrix}1\\1\end{bmatrix}\right|}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}.$$

□

6. (20 points) Consider the function

$$\mathbf{h}(x,y) = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

(a) **(10 points)** Find the Jacobi matrix of $\mathbf{h}$.
(b) **(10 points)** Find a linear approximation of $\mathbf{h}$ at $(1,1)$.

Proof. (a) The Jacobi matrix of $\mathbf{h}$ is equal to

$$D\mathbf{h}(x,y) = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}$$

where $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ and $g(x,y) = \frac{y}{\sqrt{x^2+y^2}}$. Direct computation shows

$$\begin{aligned} f_x(x,y) &= \frac{y^2}{(x^2+y^2)^{3/2}}, \quad & f_y(x,y) &= \frac{-xy}{(x^2+y^2)^{3/2}}, \\ g_x(x,y) &= \frac{-xy}{(x^2+y^2)^{3/2}}, \quad & g_y(x,y) &= \frac{x^2}{(x^2+y^2)^{3/2}}. \end{aligned}$$

Hence

$$D\mathbf{h}(x,y) = \frac{1}{(x^2+y^2)^{3/2}} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}.$$

(b) The linear approximation of $\mathbf{h}$ at $(1,1)$ is

$$\begin{aligned} L(x,y) &= \mathbf{h}(1,1) + D\mathbf{h}(1,1) \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2+x-y}{2\sqrt{2}} \\ \frac{2+y-x}{2\sqrt{2}} \end{bmatrix}. \end{aligned}$$

□