

①

$$\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 5 & 2 & 3 & 2 \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & -8 & 3 & -3 \end{array} \begin{array}{l} R_4 \\ R_5 \\ R_6 \end{array}$$

$$\begin{array}{ccc|c} R_4 & 1 & 2 & 0 \\ -R_5 & 0 & 1 & -1 \\ R_6 - 8R_5 & 0 & 0 & -5 \end{array} \begin{array}{l} 1 \\ 2 \\ 13 \end{array}$$

$$x + 2y = 1$$

$$y - z = 2$$

$$-5z = 13$$

$$x = 1 - 2y = 1 + \frac{6}{5} = \frac{11}{5}$$

$$y = 2 + z = 2 - \frac{13}{5} = -\frac{3}{5}$$

$$z = -\frac{13}{5}$$

$$(x, y, z) = \left(\frac{11}{5}, -\frac{3}{5}, -\frac{13}{5} \right)$$

②

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 \\ 2 & 5-\lambda \end{bmatrix} = (1-\lambda)(5-\lambda) = 0$$

• Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 5$

• Eigen vectors:

1). for $\lambda_1 = 1$: $\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{cases} x = x \\ 2x + 5y = y \end{cases} \Leftrightarrow 2x = -4y \Leftrightarrow x = -2y$$

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

2). for $\lambda_2 = 5$

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix}$$

$$\begin{cases} x = 5x \\ 2x + 5y = 5y \end{cases} \Leftrightarrow x = 0$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

• Express $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a\vec{v}_1 + b\vec{v}_2 = \begin{bmatrix} -2a \\ a+b \end{bmatrix}$

$$\begin{cases} -2a = 1 \\ a+b = 2 \end{cases} \Leftrightarrow \begin{cases} a = -\frac{1}{2} \\ b = 2 - a = 2 + \frac{1}{2} = \frac{5}{2} \end{cases}; \boxed{\vec{x} = -\frac{1}{2}\vec{v}_1 + \frac{5}{2}\vec{v}_2}$$

• $A^{20}\vec{x} = -\frac{1}{2}A^{20}\vec{v}_1 + \frac{5}{2}A^{20}\vec{v}_2 = -\frac{1}{2}\vec{v}_1 + \frac{5}{2} \cdot 5^{20}\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} + \frac{5}{2} \cdot 5^{20} \end{bmatrix}$

③

$$\det \begin{bmatrix} a & 1 \\ 2 & b \end{bmatrix} = ab - 2 = 10$$

$$M \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & 1 \\ 2 & b \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow b = 2$$

$$ab - 2 = 10 \Rightarrow a = \frac{10 + 2}{b} = 6$$

(4)

(a) Plane:

$$(x-1) \cdot 0 + (y-2) \cdot 1 + (z-3) \cdot 2 = 0$$

$$y - 2 + 2z - 6 = 0$$

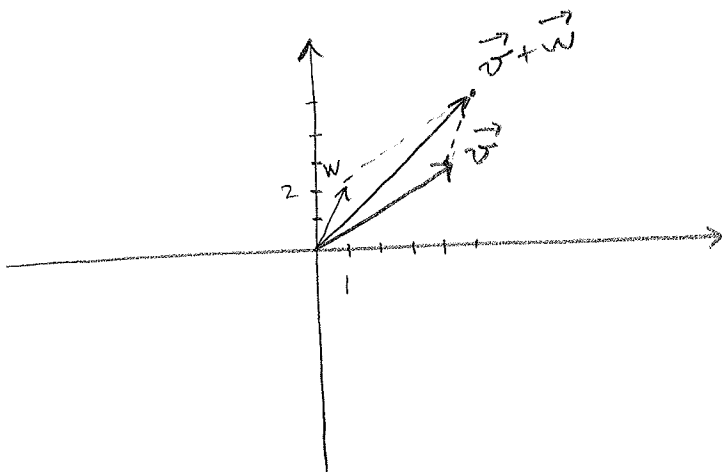
$$\boxed{y + 2z - 8 = 0}$$

(b) Line: $\vec{OP} - \vec{OQ} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -6 \end{bmatrix}$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -6 \end{bmatrix}}$$

5)

$$(a) \quad \vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$$(b) \quad \cos \theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|} = \frac{10}{5\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$|\vec{v}| = \sqrt{16 + 9} = 5$$

$$|\vec{w}| = \sqrt{1 + 4} = \sqrt{5}$$

$$\vec{v} \cdot \vec{w} = 4 + 6 = 10$$

$$\theta = \arccos \frac{2}{\sqrt{5}}$$

$$(c) \quad \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\alpha = \frac{\pi}{2}$$

$$(d) \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$