Test 1 Version 2

① $$\frac{du}{dt} = -u\sin(t), \quad u\left(\frac{\pi}{2}\right) = 2$$

$$\int \frac{du}{u} = \int -\sin(t)\,dt$$

$$\ln|u| = \cos(t) + C$$

$$|u| = e^{\cos(t) + C}$$

$$u = \underbrace{\pm e^{C}}_{D} \cdot e^{\cos(t)}$$

$$u(t) = D \cdot e^{\cos(t)}$$

$$2 = u\left(\frac{\pi}{2}\right) = D \cdot e^{\cos\left(\frac{\pi}{2}\right)} = D \cdot e^{0} = D$$

$$\boxed{\therefore u(t) = 2e^{\cos(t)}}$$

(2)

- $$\int e^{2\cot(x)}\csc^2(x)\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C$$

$u = 2\cot(x)$

$du = -2\csc^2(x)\,dx$

$-\frac{1}{2}du = \csc^2(x)\,dx$

$$= \boxed{-\frac{1}{2}e^{2\cot(x)} + C}$$

- $$\int x^2 e^{5x}\,dx = \frac{1}{5}x^2e^{5x} - \int \frac{1}{5}e^{5x}\cdot 2x\,dx$$

$u = x^2 \qquad dv = e^{5x}\,dx$

$du = 2x\,dx \qquad v = \frac{1}{5}e^{5x}$

$$= \frac{1}{5}x^2e^{5x} - \frac{2}{5}\int x e^{5x}\,dx$$

$$= \boxed{\frac{1}{5}x^2e^{5x} - \frac{2}{5}\left(\frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x}\right) + D}$$

$\int xe^{5x}\,dx$

$u = x \qquad dv = e^{5x}\,dx$

$du = dx \qquad v = \frac{1}{5}e^{5x}$

$$\int xe^{5x}\,dx = x\cdot\frac{1}{5}e^{5x} - \int\frac{1}{5}e^{5x}\,dx$$

$$= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C$$

- $\displaystyle\int \frac{2x+2}{(x-5)(x+1)}\,dx = \int \frac{2\cancel{(x+1)}}{(x-5)\cancel{(x+1)}}\,dx$

$$= \int \frac{2}{x-5}\,dx = \boxed{2\ln|x-5| + C}$$

- $\displaystyle\int_0^2 \frac{dx}{(x-1)^3} = \lim_{z \to 1^-} \int_0^z \frac{dx}{(x-1)^3} + \lim_{z \to 1^+} \int_z^2 \frac{dx}{(x-1)^3}$

$$= \lim_{z \to 1^-} \left[\frac{-1}{2(x-1)^2}\right]_0^z + \lim_{z \to 1^+} \left[\frac{-1}{2(x-1)^2}\right]_z^2$$

$$= \lim_{z \to 1^-} \left[-\frac{1}{2(z-1)^2} + \frac{1}{2\cdot(-1)^2}\right] + \lim_{z \to 1^+} \left[-\frac{1}{2\cdot(1)^2} + \frac{1}{2(z-1)^2}\right]$$

$$= -\infty + \infty = \text{D.N.E.}$$

(3) $$\frac{dy}{dt} = y^2 - 7y + 10$$

(a) $$y^2 - 7y + 10 = 0$$

$$(y-2)(y-5) = 0$$

$\boxed{y = 2 \text{ and } y = 5 \text{ are the equilibria}}$

(b) $$g(y) := y^2 - 7y + 10$$

$$\frac{dg}{dy} = 2y - 7$$

$$\left.\frac{dg}{dy}\right|_{y=2} = 4 - 7 = -3 < 0 \Rightarrow \boxed{y=2 \text{ is locally stable}}$$

$$\left.\frac{dg}{dy}\right|_{y=5} = 10 - 7 = 3 > 0 \Rightarrow \boxed{y = 5 \text{ is unstable}}$$

(4)

$$\begin{cases} x + y - z = 0 & (R_1) \\ 2x + 3y + z = 2 & (R_2) \\ y - z = 3 & (R_3) \end{cases}$$

$$\begin{array}{l} (R_1) \\ -2(R_1)+(R_2) \\ (R_3) \end{array} \begin{cases} x + y - z = 0 & (R_4) \\ y + 3z = 2 & (R_5) \\ y - z = 3 & (R_6) \end{cases}$$

$$\begin{array}{l} (R_4) \\ (R_5) \\ (R_6)-(R_5) \end{array} \begin{cases} x + y - z = 0 \\ y + 3z = 2 \\ -4z = 1 \end{cases}$$

$\boxed{z = -\frac{1}{4}}$ ✓ $\boxed{y} = 2 - 3z = 2 + \frac{3}{4} = \boxed{\frac{11}{4}}$ ✓

✓ $\boxed{x} = -y + z = -\frac{11}{4} - \frac{1}{4} = -\frac{12}{4} = \boxed{-3}$ ✓

Unique solution:

$$\boxed{(x, y, z) = \left(-3, \frac{11}{4}, -\frac{1}{4}\right)}$$

5 (a) Concentration at time $t = 0$

$$\frac{5}{100} = 0.05 \ \frac{kg}{liter}$$

(b) From part (a) $C(0) = 0.05$

$$\frac{dC}{dt} = 0.5(220 - C)$$

$$\int \frac{dC}{220 - C} = \int 0.5\,dt$$

$$-\ln|220 - C| = 0.5t + D$$

$$\ln|220 - C| = -0.5t - D$$

$$|220 - C| = e^{-0.5t - D}$$

$$220 - C = \underbrace{\pm e^{-D}}_{A} \cdot e^{-0.5t}$$

$$C(t) = 220 - A \cdot e^{-0.5t}$$

$$0.05 = C(0) = 220 - A \cdot e^{0} = 220 - A$$

$$A = 220 - 0.05 = 219.95$$

$$\boxed{\therefore \ C(t) = 220 - 219.95\,e^{-0.05t}}$$