110.107 Calculus II
Fall 2012
Exam 2 Practice

Problem 1 *Show that following limits does not exist.*

(a) $\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y}$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{x^2+y^2}$

Solution: (a) First, we consider the line $y = x$ and compute the limit along this line:

$$\lim_{x\to 0} \frac{2x}{x^2+x} = \lim_{x\to 0} \frac{2}{x+1} = 2$$

On the other hand, considering the curve $y = x^2$ we see that

$$\lim_{x\to 0} \frac{x+x^2}{2x^2} = \lim_{x\to 0} \frac{1+x}{2x} = \frac{1}{2}$$

therefore, the limit does not exist.

(b) Again computing the limit along $y = 0$ we get

$$\lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

On the other hand, computing the limit along $y = -x$ we get

$$\lim_{x\to 0} \frac{x^2 - x^2}{2x^2} = 0$$

hence, the limit does not exist.

Problem 2 ***Let*** $f(x, y) = x^y$ ***for*** $x > 0, y > 0$. ***Compute*** $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$.

Solution

$$\frac{\partial f}{\partial x} = yx^{y-1}$$
$$\frac{\partial f}{\partial y} = ln(x)x^y$$
$$\frac{\partial^2 f}{\partial x^2} = (y^2 - y)x^{y-2}$$
$$\frac{\partial^2 f}{\partial y^2} = (ln(x))^2 x^y$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{x}x^y + ln(x)yx^{y-1} = x^{y-1} + yln(x)x^{y-1}$$

Problem 3 ***Let*** $G(x,x) = x^2 + xy^2 - \frac{y^2}{2}$ ***and*** $p = (-1,1) \in \mathbb{R}^2$***.***

(a) Calculate the equation of the plane tangent to the graph of G ***at the point*** p.

(b) Calculate the gradient of G ***at the point*** p ***(that is, compute*** $\nabla G(-1,1)$***).***

(c) Find a critical point of $G(x,y)$ ***on the domain*** $\mathbb{R}^2$ ***and determine whether it is a local maximum, local minimum, or neither (Hint: The Hessian will help here.)***

Solution (a) The equation of the tangent plane to the graph of G **at** p **is of the form**

$$z - z_0 = A(x - x_0) + B(y - y_0)$$

where $A = \frac{\partial G}{\partial x}(p)$ **and** $B = \frac{\partial G}{\partial y}(p)$ **and** $z_0 = G(p)$. **Computing these values:**

$$z_0 = G(-1,1) = -\frac{1}{2}$$
$$A = \frac{\partial G}{\partial x}(-1,1) = (2x + y^2)|_{x=-1,y=1} = -1$$
$$B = \frac{\partial G}{\partial y}(-1,1) = (2xy - y)|_{x=-1,y=1} = -3$$

Thus, the equation of the plane is given by

$$z + \frac{1}{2} = -(x+1) - 3(y-1).$$

(b)The Gradient of G is given by

$$\nabla G(-1,1) = \left[\begin{array}{c} \frac{\partial G}{\partial x} \\ \frac{\partial G}{\partial y} \end{array} \right]_{x=-1,y=1} = \left[\begin{array}{c} -1 \\ -3 \end{array} \right]$$

(c) To find the critical point(s) we need to solve

$$\nabla G(x,y) = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

That is

$$2x + y^2 = 0$$
$$y(2x-1) = 0$$

The second equation implies that either $y = 0$ or $2x - 1 = 0$. If $y = 0$ then $x = 0$ by the first equation. If $2x - 1 = 0$ then $x = \frac{1}{2}$ and from the first equation $y^2 = -1$ which has no real solutions. Thus the function $G(x,y)$ has only one critical point in $\mathbb{R}^2$ which is $(0,0)$.

Now, we apply the second derivative test:

$$Hes(G)(x,y) = \left[\begin{array}{cc} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{array} \right]_{x=0,y=0} = \left[\begin{array}{cc} 2 & 2y \\ 2y & 2x-1 \end{array} \right]_{x=0,y=0} = \left[\begin{array}{cc} 2 & 0 \\ 0 & -1 \end{array} \right]$$

Since the determinant of the Hessian $D = -2 < 0$ by second derivative test the point $(0,0)$ is a saddle point i.e. it is neither a local min or max.

Problem 4 ***Given the system***

$$\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 4x - 2y \end{aligned}$$

(a) *Solve the system for the particular solution that passes through the point* $(x, y) = (1, 0)$.

(b) *Find all equilibrium solutions and determine their stability.*

(c) *Draw the solution passing through the point* $(x, y) = (1, 0)$ *on the direction field for all* $t \in \mathbb{R}$. *Also draw the solution passing through the point* $(x, y) = (1, 1)$ *for all* $t \in \mathbb{R}$. *(Use the Java applet to produce the direction filed)*

Solution (a) We rewrite the system in the matrix from

$$\frac{d\vec{x}}{dt} = A\vec{x} \textbf{ where } A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \textbf{ and } \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Given this system, we can write the general solutions as follows: If A has two distinct real eigenvalues λ_1, λ_2 and $\vec{u}_1, \vec{u}_2$ are the corresponding eigenvectors then the general solution to the above system is given by

$$\vec{x}(t) = c_1\vec{u}_1e^{\lambda_1 t} + c_2\vec{u}_2e^{\lambda_2 t}.$$

To find eigenvalues we solve the characteristic equation $det(\lambda I_2 - A) = 0$ which is $\lambda^2 + \lambda - 6 = 0$. Thus, eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -3$. For the eigenvalue $\lambda = 2$ the eigenvector equation $A\vec{u} = 2\vec{u}$ leads to the system $x + y = 2x$ and $4x - 2y = 2y$. Remember these two equations are ALWAYS the same equation when finding the eigenvectors, and any vector $\vec{u}$ that satisfies either equation one works. The first equation leads directly to $x = y$. Choose $x = 1$, so that $y = 1$, and an eigenvector for $\lambda_1 = 2$ is $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If we do the same thing for $\lambda_2 = -3$, we will get the equation $x + y = -3x$, and if we choose $x = 1$, we get $y = -4$, and for $\lambda_2 = -3$, we get $\vec{u}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$. Hence the general solution to this system is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$$

For our particular solution, we have a starting point $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Throw that into the general solution evaluated at $t = 0$, to get the

values of the two unknown constants c_1 and c_2 that correspond to the solution that passes through the point $(1, 0)$. Hence,

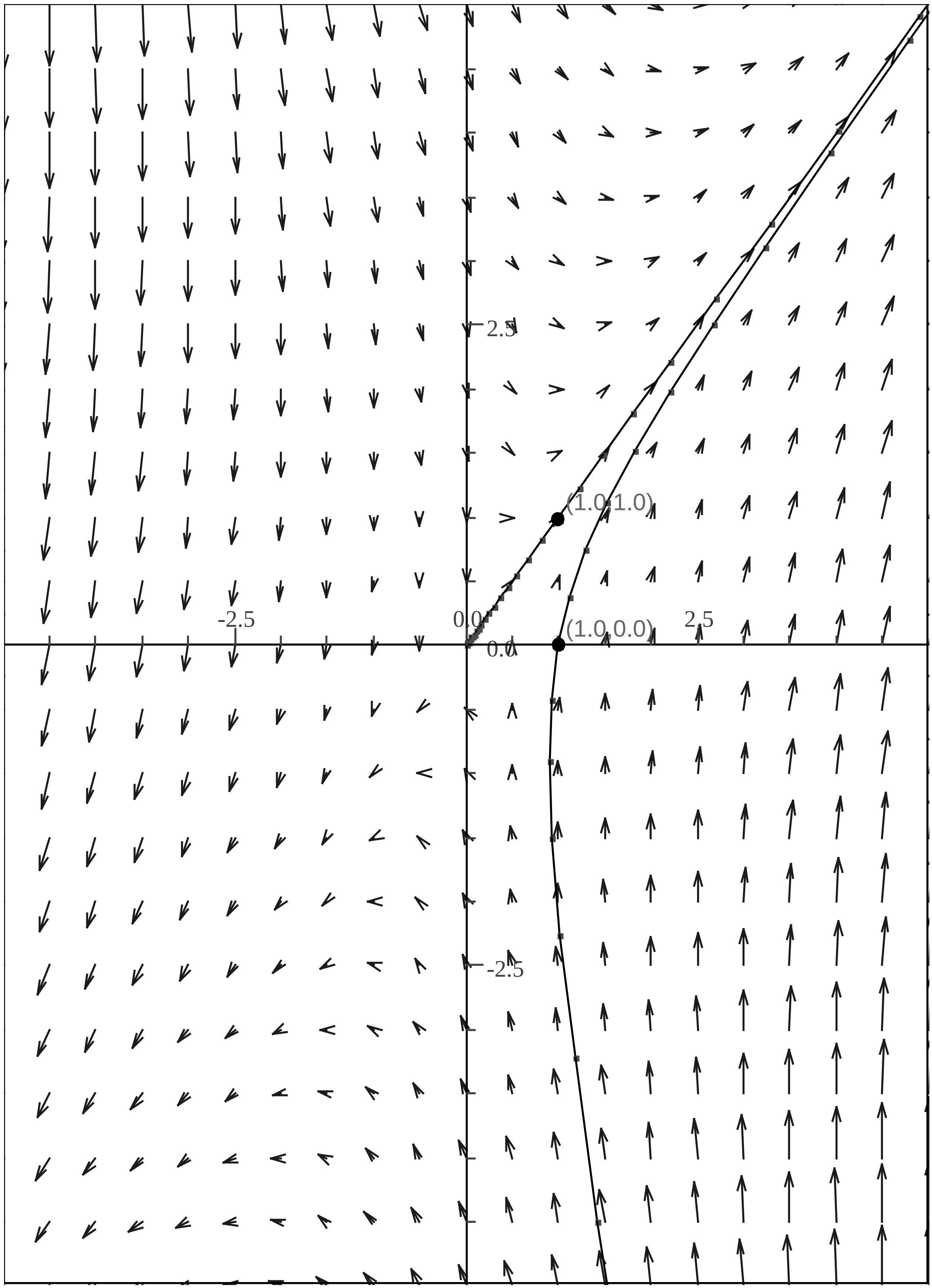
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

This leads to the two equations $1 = c_1 + c_2$ and $0 = c_1 - 4c_2$. Solving these two leads to $c_1 = \frac{4}{5}$ and $c_2 = \frac{1}{5}$. Hence our particular solution is

$$\vec{x}(t) = \frac{4}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \frac{1}{5} \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$$

(b) The only equilibrium of a linear system where the matrix A has non-zero determinant (like this one) is the origin. And since the two eigenvalues here are real, distinct, non-zero, and of different signs. Thus, the origin is a saddle and unstable.

(c) On the next page, the solution passing through the point $(x, y) = (1, 0)$ is drawn on the direction field for all $t \in \mathbb{R}$. The solution passing through the point $(x, y) = (1, 1)$ is also drawn. Note that these two points are marked.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