Fall 2012 110.107 Calc II Exam 1

October 15, 2012

1. [10 pts]

Let $A = \begin{bmatrix} -1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & -1 \end{bmatrix}$.

Decide which of the following operations are well defined, and carry out those, which are defined. If not defined, briefly explain the reason.

a) $B\,(C - 2A)$

$$B\,(C-2A) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & -1 \end{bmatrix} - 2 \begin{bmatrix} -1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & -1 \end{bmatrix} - \begin{bmatrix} 2\,(-1) & 2\,(0) & 2\,(3) \\ 2\,(1) & 2\,(-2) & 2\,(1) \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 6 \\ 2 & -4 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2-(-2) & 1-0 & 0-6 \\ 0-2 & -3-(-4) & -1-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -6 \\ -2 & 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\,(4)+(-1)\,(-2) & 1\,(1)+(-1)\,(1) & 1\,(-6)+(-1)\,(-3) \\ 0\,(4)+1\,(-2) & 0\,(1)+1\,(1) & 0\,(-6)+1\,(-3) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 6 & 0 & -3 \\ -2 & 1 & -3 \end{bmatrix}}$$

b) $C\,(A+B)'$

$A+B$ would be $\begin{bmatrix} -1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, which is the sum of two matrices with different dimensions. Thus, the operation $A+B$ is not defined. Since it's not defined, we certainly can't take the transpose of it and then left-multiply by C, so the whole expression is $\boxed{\text{not defined}}$.

2. [15 pts]

Consider the following system of linear equations

$$x - y + 2z = 2$$

$$x + 2y - z = 5$$

$$y - z = 1$$

a) Write the augmented matrix of the system.

$$\boxed{\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 1 & 2 & -1 & 5 \\ 0 & 1 & -1 & 1 \end{array}\right]}$$

b) Find the solution set of the system.

Subtracting the first row from the second (and writing in the second row) yields: $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & -3 & 3 \\ 0 & 1 & -1 & 1 \end{array}\right]$. Subtracting three times the third row from the second (and writing in the third row) yields $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Now that the system in upper-triangular form, we can see that it is underdetermined. We may let z be an arbitrary parameter t. The second row represents $3y - 3z = 3$, so $y = z + 1 = t + 1$. Finally, the first equation represents

$$x - y + 2z = 2, \text{ so } x - (t+1) + 2t = 2, \text{ and}$$

$$x - t - 1 + 2t = 2 \text{ so that } x + t - 1 = 2 \text{ and } x = 3 - t.$$

Thus, the solution set is $\boxed{\{(3-t, t+1, t) : t \in \mathbb{R}\}}$.

Note: There are many other valid ways of writing the solution set.

2. [20 pts]

Each of the two definite integral below is improper. Choose ONLY ONE of them. For this choice, either show that the integral does not exist, or calculate its value.

a)

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x+1}} \mathrm{d}x$$

First, note that while the bounds of integration are finite, the integrand is undefined at -1 (due to division by zero). Therefore, we need to write the integral as a limit:

$$\lim_{a \to -1^+} \int_{a}^{1} \frac{1}{\sqrt[3]{x+1}} \mathrm{d}x$$

We then substitute $u = x + 1$, $\mathrm{d}u = \mathrm{d}x$. So that we don't have to worry about converting things back in terms of x at the end, we change the bounds of integration now: If $x = a$, then $u = a + 1$, and if $x = 1$, then $u = 1 + 1 = 2$, so the integral becomes:

$$\lim_{a \to -1^+} \int_{a+1}^{2} \frac{1}{\sqrt[3]{u}} \mathrm{d}u = \lim_{a \to -1^+} \int_{a+1}^{2} u^{-1/3} \mathrm{d}u$$

$$= \lim_{a \to -1^+} \left(\frac{3}{2} u^{2/3} \Big|_{a+1}^{2} \right) = \lim_{a \to -1^+} \left(\frac{3}{2} * 2^{2/3} - \frac{3}{2} (a+1)^{2/3} \right)$$

$$= \frac{3}{2} * 2^{2/3} - \frac{3}{2} (-1+1)^{2/3} \text{ since } f(x) = (x+1)^{2/3} \text{ is continuous everywhere}$$

$$= \boxed{\frac{3}{2} * \sqrt[3]{4}}.$$

b)

$$\int_{2}^{\infty} \frac{x}{(x^2-1)^2} \mathrm{d}x$$

First, note that the integrand is defined everywhere except $x = \pm 1$, so that the only limit we have to worry about is the one for the infinite bound of integration:

$$\lim_{b \to \infty} \int_{2}^{b} \frac{x}{(x^2-1)^2} \mathrm{d}x.$$

We then substitute $u = x^2 - 1$, $\mathrm{d}u = 2x\mathrm{d}x$, so $(1/2)\,\mathrm{d}u = x\mathrm{d}x$. So that we don't have to worry about converting things back in terms of x at the end, we change the bounds of integration now: If $x = 2$, then $u = 2^2 - 1 = 3$, and if $x = b$, then $u = b^2 - 1$, so the integral becomes:

$$\lim_{b \to \infty} \int_{3}^{b^2-1} \frac{(1/2)\,\mathrm{d}u}{u^2} = \lim_{b \to \infty} \frac{1}{2} \int_{3}^{b^2-1} \frac{\mathrm{d}u}{u^2}$$

$$= \lim_{b \to \infty} \frac{1}{2} \int_{3}^{b^2-1} u^{-2} \mathrm{d}u = \lim_{b \to \infty} \frac{1}{2} \left(-u^{-1} \Big|_{3}^{b^2-1} \right)$$

$$= \lim_{b\to\infty} \frac{1}{2}\left(-\frac{1}{b^2-1} - \left(-\frac{1}{3}\right)\right) = \frac{1}{6} - \lim_{b\to\infty} \frac{1}{2\left(b^2-1\right)} = \boxed{\frac{1}{6}}$$

with the final step following from the fact that $2\left(b^2-1\right)$ grows without bound as b does, so that the limit of the whole fraction is zero.

4. [20 pts]

Solve the following first order seperable initial value problem for $x \geq 0$ given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-y}\left(2+\cos(x)\right) \text{ and } y(0) = 0.$$

$$\text{We have } \frac{\mathrm{d}y}{e^{-y}} = \left(2+\cos(x)\right)\mathrm{d}x \text{ (this division is ok since } e^{-y} \text{ is never 0)}$$

$$\Rightarrow \int \frac{\mathrm{d}y}{e^{-y}} = \int \left(2+\cos(x)\right)\mathrm{d}x \Rightarrow \int e^{y}\mathrm{d}y = \int \left(2+\cos(x)\right)\mathrm{d}x$$

$$\Rightarrow e^{y} = 2x + \sin x + C$$

$$\Rightarrow y = \ln\left(2x+\sin x + C\right) \text{ is the general solution}$$

Now we must use the initial value, plugging in 0 for y and 0 for x since $y(0) = 0$:

$$0 = \ln\left(2(0) + \sin 0 + C\right) = \ln(C), \text{ so } C = 1$$

Thus, the particular solution to this initial value problem is

$$\boxed{y = \ln\left(2x + \sin x + 1\right)}.$$

5. [10 pts]

Consider the line in the plane passing through the point $(-1, 2)$ and perpendicular to the vector $\vec{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a) Find an equation for the line.

Reading off the numerical data, one equation is $3\left(x-(-1)\right) + 4\left(y-2\right) = 0$, i.e. $\boxed{3(x+1) + 4(y-2) = 0}$ or $\boxed{3x + 4y - 5 = 0}$. The first equation can come from a memorized formula from your textbook.

However, if you didn't memorize that, note that $\begin{bmatrix} x-(-1) \\ y-2 \end{bmatrix}$ is the vector (in the direction of the line) from the point $(-1, 2)$ to an arbitrary point on the line (x, y). Then since the line is perpendicular to $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, we have

$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} x-(-1) \\ y-2 \end{bmatrix} = 0$, so $3\left(x-(-1)\right) + 4\left(y-2\right) = 0$✓

b) Find a parametric equation for the line.

Now that we have an equation, we can let almost any expression be our parameter; let's take $x = t$ for simplicity. Then $3t + 4y - 5 = 0$ so $y = (5 - 3t)/4$.

$\boxed{x = t \text{ and } y = (5-3t)/4}$ is one way to write the answer.

6. [25 pts]

Let $A = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ and assume that $A \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

a) Find x and y.

$$A \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x & y \\ y & x \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x*3+y*4 \\ y*3+x*4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 3x+4y &= 5 \\ 4x+3y &= 2 \end{cases}$$

$$\Rightarrow \begin{cases} x-y &= 2-5 \\ 4x+3y &= 2 \end{cases} \text{ by subtracting the first from the second}$$

$$\Rightarrow \begin{cases} x &= y-3 \\ 4x+3y &= 2 \end{cases} \Rightarrow \begin{cases} x &= y-3 \\ 4(y-3)+3y &= 2 \end{cases}$$

$$\Rightarrow \begin{cases} x &= y-3 \\ 7y-12 &= 2 \end{cases} \Rightarrow \begin{cases} x &= y-3 \\ 7y &= 14 \end{cases}$$

$$\Rightarrow \begin{cases} x &= y-3 \\ y &= 2 \end{cases} \Rightarrow \boxed{\begin{cases} x &= -1 \\ y &= 2 \end{cases}}$$

b) Calculate A^{-1}

From part a), $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$, so $\det A = (-1)(-1) - (2)(2) = -3$.

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}}$$

c) Calculate the eigenvalues and corresponding eigenvectors of A.

First, we must find the eigenvalues, using $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}$$

$$\text{so } \det(A - \lambda I) = (-1-\lambda)^2 - 2^2 = \lambda^2 + 2\lambda + 1 - 4 = (\lambda + 3)(\lambda - 1).$$

The eigenvalues, solutions to $\det(A - \lambda I) = 0$, are thus $\boxed{\lambda = -3}$ and $\boxed{\lambda = 1}$.
Now we must find eigenvectors:

For $\lambda = -3$, an eigenvector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ satisfies

$$A\vec{v} = -3\vec{v} \text{ so } \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -3\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -v_1 + 2v_2 \\ 2v_1 - v_2 \end{bmatrix} = \begin{bmatrix} -3v_1 \\ -3v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2v_1 + 2v_2 \\ 2v_1 + 2v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_2 = -v_1.$$

Thus, every eigenvector corresponding to the eigenvalue $\lambda = -3$ has the form $\begin{bmatrix} v_1 \\ -v_1 \end{bmatrix}$ for some nonzero v_1. However, full credit was given for a particular eigenvector, such as $\boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$.

For $\lambda = 1$, an eigenvector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ satisfies

$$A\vec{v} = 1\vec{v} \text{ so } \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 1\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -v_1 + 2v_2 \\ 2v_1 - v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -2v_1 + 2v_2 \\ 2v_1 - 2v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_2 = v_1.$$

Thus, every eigenvector corresponding to the eigenvalue $\lambda = 1$ has the form $\begin{bmatrix} v_1 \\ v_1 \end{bmatrix}$ for some nonzero v_1. However, full credit was given for a particular eigenvector, such as $\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.