Solutions to Final Exam Review for Math 107

1a. Set $u = \ln(1+x)$, $dv = x^2$.

$$\int_0^1 x^2 \ln(1+x)\ dx = \ln(1+x)\frac{x^3}{3}|_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{1+x}\ dx\ .$$

By long division,

$$\frac{x^3}{1+x} = (x^2 - x + 1) - \frac{1}{x+1}\ .$$

Hence,

$$\int_0^1 x^2 \ln(1+x)\ dx = \frac{1}{3}\ln 2 - \int_0^1 [(x^2 - x + 1) - \frac{1}{x+1}]\ dx = \frac{2}{3}\ln 2 - \frac{5}{18}$$

b. Using long division again and partial fractions,

$$\int \frac{x^3}{(x^2-1)}\ dx = \int (x + \frac{x}{(x-1)(x+1)}\ dx = \int (x + \frac{1}{2}(\frac{1}{x-1} + \frac{1}{x+1})\ dx = \frac{x^2}{2} + \frac{1}{2}\ln(x^2-1) + c\ .$$

c. Use integration by parts with $u = \frac{1}{2\pi}x^2$, $dv = 2\pi x \cos \pi x^2$. Then

$$\int_0^1 x^3 \cos(\pi x^2)\ dx = \frac{1}{2\pi}x^2 \sin \pi x^2|_0^1 - \frac{1}{\pi}\int_0^1 x \sin \pi x^2\ dx = \frac{2}{\pi^2}\cos \pi x^2|_0^1 = -\frac{1}{\pi^2}\ .$$

2. $y' = 0.5(y + \frac{3}{5})$, $y(0) = 0.8$ so,

$$y + \frac{3}{5} = C \exp(.5t)$$

and $C = 1.4$. Therefore, $y = -.6 + (1.4)\exp(.5t)$ a. $y(1) = -.6 + (1.4)\exp .5$
b. $e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + R_n(x)$ with

$$|R_n| \leq 3\frac{|x|^{n+1}}{(n+1)!}$$

We want to choose n so that $|R_n| \leq (10)^{-6}$ when $x = .5$. That is, we want

$$3\frac{2^{(n+1)}(n+1)!}{\leq}(10)^{-6}$$

or

$$2^{(n+1)}(n+1)! \geq 3(10)^{-6}\ .$$

Using a calculator we see $n = 7$ is needed. This gives

$e^{.5} \approx 1+.5+.125+.02083333+.00260416+.000260416+.00002170138.00000155 = 1.64872127$

and $y(1) \approx 1.7082098$.

3. The total number of people infected with a virus often grows like a logistic curve. Suppose that 10 people originally have the virus, and that in the early stages of the virus (with time measured in weeks), the number of people infected is increasing exponentially with k=1.35. It is estimated that, in the long run, approximately 5000 people become infected.

a) $M = 5000$, $k = 1.35$ and $y(0) = 10$. Hence

$$y' = (1.35)y\frac{(5000-y)}{5000} .$$

$$5000\int \frac{dy}{y(5000-y)} = (1.35)t + c$$

$$\int (\frac{1}{y} + \frac{1}{5000-y})\ dy = (1.35)t + c$$

$$\ln|\frac{y}{5000-y}| = 1.35t + c$$

$$\frac{y}{5000-y} = C\exp 1.35t$$

See my tutorial on the Logistic equation. Putting $t = 0$ gives $C = \frac{1}{499}$
b. Solve $y(t) = 2500$ or equivalently

$$1 = \frac{1}{499}\exp 1.35t$$

$\exp 1.35t = 499$, $1.35t = \ln 499 = 6.212606$, $t = 4.60$ weeks

4. (15 pts) Approximate $\int_0^{.2} \sin(x^3)\ dx$ with a error less than 0.0001 . Justify the error estimate.

5.

$$\int_0^\infty \frac{dx}{x^2+3x+2} = \lim_{L\to\infty}\int_0^L \frac{1}{(x+1)(x+2)}$$

$$= \lim_{L\to\infty} \int_0^L (\frac{1}{x+1} - \frac{1}{x+2})\ dx =$$

$$= \lim_{L\to\infty} \ln \frac{x+1}{x+2}|_0^L = \lim_{L\to\infty} (\ln \frac{L+1}{L+2} + \ln 2) = \ln 2$$

6. (15 pts) The lifespan of a light bulb is exponentially distributed with a mean of 100 hours. Since $\mu = \frac{1}{k}$, $k = .01$ and

$$p(x) = .01 \exp -.01x \ , \ F(x) = 1 - \exp -.01x$$

The probability that a light bulb will last at least 75 hours is $\exp -(.01)(75) = \exp -.75 = 0.4724$

7. (15 pts) Let A= (4,3), B=(1, -2)
Using vectors, find the distance from B to the line in the direction of $\vec{OA}$.

First find $\vec{OB}_{\vec{OA}} = \frac{1}{25}(\vec{i} - 2\vec{j}) \cdot (4\vec{i} + 3\vec{j})(4\vec{i} + 3\vec{j}) = -\frac{2}{25}(4\vec{i} + 3\vec{j})$
Then

$$|vecOB - \vec{OB}_{\vec{OA}}| = |(\vec{i} - 2\vec{j}) + \frac{2}{25}(4\vec{i} + 3\vec{j})| = |\frac{1}{25}(33\vec{i} - 44\vec{j})| = |\frac{11}{25}(3\vec{i} - 4\vec{j}) = \frac{11}{5}$$

is the distance.

8. $\vec{r}(t) = \ln t\ \vec{i} + t\ \vec{j}$ At $t = 1$, $\vec{v} = \vec{r}'(1) = \vec{i} + \vec{j}$. Hence the equation of the tangent line is

$$\vec{r}(t) = \vec{j} + t(\vec{i} + \vec{j})$$

9. $f(x,y) = x^4 + xy^2$. The normal direction to the level set $f(x,y) = -2$ at the point (-2,3) is in the direction $\nabla f(-2,3) = -23\vec{i} - 12\vec{j}$ so

$$\vec{r}(t) = -2\vec{i} + 3\vec{j} + t(-23\vec{i} - 12\vec{j})$$

is the normal line.

10. Take $x_0 = 0$, $y_0 = 20,000$, $x'(0) = 250$, $y'(0) = 0$. The equations of motion are

$$x''(t) = 0 \ , \ x'(t) = 250 \ , \ x(t) = 250t$$

$$y''(t) = -9.8 \ , \ y'(t) = -9.8t \ , \ y(t) = -4.9t^2 + 20,000$$

The engine hits the ground when $y(t) = 0$ or $4.9t^2 = 20,000$ or $t = 63.8876$ seconds later. It has travelled horizontally a distance of $250(63.8876) =$

$15,972$ meters.

11. a. $h_x = 4x$, $h_y = 6y$ so $\nabla h(-1,2) = -4\vec{i} + 12\vec{j}$. We want $D_{\vec{a}}h(-1,2) = \nabla h(-1,2) \cdot \vec{a} = (-4\vec{i} + 12\vec{j}) \cdot (\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}) = \frac{20}{5} = 4$.

b. The tangent direction to the contour line of $h(x,y)$ through (-1,2) is perpendicular to $\nabla h(-1,2)$. So we need to find a vector perpendicular to $-4\vec{i} + 12\vec{j}$. We can take $\pm\frac{1}{\sqrt{10}}(3\vec{i} + \vec{j})$.

c. The depth decreases the fastest in the direction of $-\nabla h$ or in the direction $\frac{1}{\sqrt{10}}(\vec{i} - 3\vec{j})$.