Math 107 Final Exam and Solutions May 10, 2001 Professor Spruck

1a. (15 pts) Evaluate $\int x^{-2} \ln(1+x)\, dx$

Solution. Let $u = \ln(1+x)$, $dv = x^{-2}$ so $du = \frac{1}{1+x}dx$, $v = -x^{-1}$. Then

$$\int x^{-2} \ln(1+x)\, dx = -\frac{\ln(1+x)}{x} + \int \frac{1}{x(1+x)}\, dx = -\frac{\ln(1+x)}{x} + \int (\frac{1}{x} - \frac{1}{x+1})\, dx$$

$$= -\frac{\ln(1+x)}{x} + \ln|\frac{x}{x+1}| + C.$$

b. (15 pts) Evaluate $\int_3^\infty \frac{1}{x^2-x-2}\, dx$

Solution. Use partial fractions $\frac{1}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}$ to find $A = \frac{1}{3}$ and $B = -\frac{1}{3}$. Hence,

$$\int_3^\infty \frac{1}{x^2-x-2}\, dx = \lim_{L\to\infty} \frac{1}{3} \ln \frac{x-2}{x+1}|_3^L = \frac{1}{3}\ln 4$$

c. (15 pts) Assume that a random variable X has a normal distribution with mean 12 and standard deviation 4. Find the probability that $5.6 < X < 19.2$ using the Z-table at the end of your exam paper.

Solution. Change to standard Normal:

$$Z = \frac{X-\mu}{\sigma} = \frac{X-12}{4} = \frac{X}{4} - 3 \ .$$

$$X = 5.6\ ,\ Z = 1.4 - 3 = -1.6\ .$$

$$X = 19.2\ ,\ Z = 4.8 - 3 = 1.8\ .$$

Using the Z-table,

$$P\{-1.6 < Z < 1.8\} = F(1.8) - F(-1.6) = F(1.8) - (1 - F(-1.6))$$

$$= F(1.8) + F(1.6) - 1 = 0.9641 + 0.9452 - 1 = 0.9093\ .$$

2. (20 pts) Let $y'(t) = 10 - y(t)$, $y(0) = 0$. Find $y(t)$ and show that its asymptotic limit is 10 as $t \to \infty$. How long does it take for y(t) to reach 87.5% of its asymptotic limit? Justify your answer.

Solution. $y' = -(y - 10)$, $y - 10 = Ce^{-t}$, $C = -10$. So $y = 10(1 - e^{-t})$ and $\lim_{t\to\infty} y(t) = 10$, the asymptotic limit. We find t so that

$$10(1 - e^{-t} = \frac{7}{8} \cdot 10 \quad \text{or} \quad 1 - e^{-t} = \frac{7}{8} \ .$$

This gives $e^{-t} = 1 - \frac{7}{8} = \frac{1}{8}$. Hence $t = \ln 8 = 3 \ln 2$.

3a. (15 pts) Find a Taylor polynomial p(x) whose degree is as small as possible so that $|p(x) - \cos x| < 0.01$ for all x in the interval $[-1.5, 1.5]$. Justify your work.

Solution. Expand $\cos x$ as a Taylor polynomial of degree 2n with remainder:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots \pm \frac{x^{2n}}{(2n)!} + R_n(x) \ ,$$

where $|R_n(x)| \le \frac{|x|^{2n+2}}{(2n+2)!} \le \frac{(1.5)^{2n+2}}{(2n+2)!}$. So we need to choose the smallest n so that

$$\frac{(1.5)^{2n+2}}{(2n+2)!} < 0.01 \ .$$

$n = 2$ $\frac{(1.5)^6}{6!} = .0156$, too big.
$n = 3$ $\frac{(1.5)^8}{8!} = .000635 < .01$. Hence, $p(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$.

b. (15 pts) Approximate $\int_0^1 e^{-(x^2)}\, dx$ with an error less than 10^{-3}. Justify your work.

Solution. Expand $\exp -x$ as a taylor polynomial of degree n with remainder:

$$\exp x = 1 - x + \frac{x^2}{2!} - \ldots \pm \frac{x^n}{n!} + R_n(x) \ ,$$

where $|R_n(x)| \le 3\frac{|x|^{n+1}}{(n+1)!}$. Now replace x by x^2 and note $|R_n(x^2)| \le 3\frac{|x|^{2n+2}}{(n+1)!}$. This gives

$$\int_0^1 e^{-x^2}\, dx = 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \ldots \pm \frac{1}{(2n+1)n!} + \qquad \text{error}$$

where error $< \frac{3}{(2n+3)(n+1)!}$. So we want to choose n so that $\frac{3}{(2n+3)(n+1)!} < 0.001$ or $(2n+3)(n+1)! > 3000$. $n = 4 \quad 11 \cdot 120 < 3000$
$n = 5 \quad 13 \cdot 720 > 3000$
This gives,

$$\int_0^1 \approx 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} - \frac{1}{11 \cdot 5!} = 0.7467.$$

4. (20 pts) Solve the differential equation $y'(t) = 2y(1 - \frac{y}{100})$; $y(0) = 5$.

Solution.

$$100 \int \frac{dy}{y(100-y)} = 2t + c$$

$$\int (\frac{1}{y} + \frac{1}{100-y})\, dy = 2t + c$$

$$\ln |\frac{y}{100-y}| = 2t + c$$

$$|\frac{y}{100-y}| = C \exp(2t) \ , \ C = \frac{5}{95} = \frac{1}{19}$$

so $\frac{y}{100-y} = \frac{1}{19} \exp(2t)$, and

$$y = \frac{100 \exp(2t)}{19 + \exp(2t)}$$

5. A cannon is sitting on a mountain top 256 ft above a road the ground. It can fire a projectile with an initial speed of 96 ft/sec but only in the horizontal direction (firing angle 0).

a. (15 pts) How long does it take for the projectile to hit the ground and how far horizontally has it traveled?

Solution. $x(t) = 96t$, $y(t) = -16t^2 + 256$. The projectile hits the ground when $y(t) = 0$, i.e. $16t^2 = 256$, $t^2 = 16$, $t = 4$ seconds after it is fired. It has traveled horizontally $x_{\max} = 96 \cdot 4 = 384$ ft.

b. (10pts) An enemy tank traveling with constant velocity 16ft/sec passes the (horizontal) position of the cannon. How many seconds must the gunner wait before firing in order to hit the tank?

Solution. The tank traveling at 16 ft/sedc takes $384/16 = 24$ sec to reach the point where the projectile lands. Since the projectile takes only 4 sec. to land. the gunner must wait 20 seconds.

6. Let $f(x,y) = x^3 - y^3$. a. (10 pts) Find the directional derivative of f(x,y) at the point (2,1) in the direction $\frac{\vec{i}-\vec{j}}{\sqrt{2}}$.

Solution. $f_x = 3x^2$, $f_y = -3y^2$; $\nabla f(2,1) = 12\vec{i} - 3\vec{j}$. With $a = \frac{\vec{i}-\vec{j}}{\sqrt{2}}$, $D_{\vec{a}} f(2,1) = \nabla f(2,1) \cdot \vec{a} = \frac{(12+3)}{\sqrt{2}} = \frac{15}{\sqrt{2}}$.

b. (10 pts) Find the equation of the tangent line to the level curve of f(x,y) passing through the point (2,1).

Solution. $\nabla f(2,1) = 12\vec{i} - 3\vec{j} = 3(4\vec{i} - \vec{j})$ is perpendicular to the contour line of f(x,y) passing through (2,1) so the direction $\vec{i} + 4\vec{j}$ is tangent to the contour. So $\vec{r}(t) = (2\vec{i} + \vec{j}) + t(\vec{i} + 4\vec{j})$ is the parametric equation of the line. Equivalently,

$$x(t) = 2 + t \ , \ y(t) = 1 + 4t$$

or eliminating $t = x - 2$, $y = 1 + 4(x-2) = 4x - 7$ is the line.

7. Let A=(1,1), B=(5,-1), C=(3,-2).

a. (10 pts) Find $\vec{AC}_{\vec{AB}}$.

Solution. $\vec{AC} = 2\vec{i} - 3\vec{j}$, $\vec{AB} = 4\vec{i} - 2\vec{j}$, $|\vec{AB}| = \sqrt{20}$, $\vec{AC} \cdot \vec{AB} = 14$. Hence,

$$\vec{AC}_{\vec{AB}} = \frac{\vec{AC} \cdot \vec{AB}}{|\vec{AB}|^2} \ \vec{AB} = \frac{14}{20}(4\vec{i} - 2\vec{j}) = \frac{7}{5}(2\vec{i} - \vec{j}) \ .$$

b. (10 pts) Find the area of triangle ABC (Hint: Use part a. to find the length of the altitude from C (the height) to the base AB; the area of any triangle is $\frac{1}{2}$ (base)(height).

Solution. The base AB has length $\sqrt{20}$ from part a. If we write

$$\vec{AP} = \vec{AC}_{\vec{AB}} = \frac{7}{5}(2\vec{i} - \vec{j}) \ ,$$

then $\vec{PC} = \vec{AC} - \vec{AP} = (2\vec{i} - 3\vec{j}) - \frac{7}{5}(2\vec{i} - \vec{j}) = -\frac{4}{5}\vec{i} - \frac{8}{5}\vec{j} = -\frac{4}{5}(\vec{i} + 2\vec{j})$, and the height $= |\vec{PC}| = \frac{4}{5}\sqrt{5}$. Hence

$$A = \frac{1}{2}\sqrt{20} \cdot \frac{4}{5}\sqrt{5} = \frac{4}{10}\sqrt{100} = 4 \ .$$

8. Suppose that a radioactive isotope has a half life of 1000 years.

a. (10pts) Find the expected lifespan of an atom.

Solution. For an exponential distribution $p(x) = ke^{-kx}$ and $F(x) = 1 - e^{-kx}$, and the mean is $\frac{1}{k}$ But $1000 = T_{\frac{1}{2}} = \frac{\ln 2}{k}$ so the expected lifespan is $\frac{1}{k} = \frac{1000}{\ln 2}$.

b. (10pts) Consider now only those atoms that last at least 1000 years. Find the expected lifespan. Hint: You must renormalize the probability density $p(t)$ to find a new probability density $\tilde{p}(t)$ for these atoms.

Solution. Note that since the half-life (median) is 1000 years, half of the original atoms die in the first 1000 years and we are interested in the other half that live at least 1000 years. We take $\tilde{p}(t) = \frac{p(t)}{\int_1 000\infty p(t)\ dt} = \frac{p(t)}{\frac{1}{2}} = 2p(t)$ as the new probability density for these atoms over the time interval $[1000, \infty)$. Then

$$\tilde{F}(t) = \int_{1000}^{t} \tilde{p}(s)\ ds = 2(F(t) - F(1000)) == 2(1 - e^{-(kt)} - \frac{1}{2}) = 1 - 2e^{-(kt)}$$

is the new cumulative distribution function. Hence the new mean lifespan is

$$\tilde{\mu} = \int_{1000}^{\infty} t\tilde{p}(t)\ dt = 2\int_{1000}^{\infty} kte^{-(kt)}\ dt$$

$$= (t(-e^{-(kt)})|_0^{\infty} + \int_{1000}^{\infty} e^{-(kt)}\ dt = 2(500 + \frac{1}{2k}) = 1000 + \frac{1}{k}\ .$$