Name: SOLUTIONS (Print)

Fall 2012

110.107 Calculus II (Biology and Social Sciences)

Final Exam

December 12, 2012

Instructions: The exam is **10** pages long, including this title page. The number of points each problem's worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck!!

	Problem	Score	Value of the problem	
m	1		10	T
S	2		10	J
L	3		10	J
L	4		10	J
L	5		15	m
L	6		15	m
S	7		10	T
L	8		10	T
S	9		10	m
	total		100	

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________ **Date:** __________

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1. [10 points] Determine if the following improper integral converges or diverges. If the integral is convergent compute its value.

$$\int_0^\infty xe^{-x}dx$$

$$\int_0^\infty xe^{-x}dx \overset{\text{def}}{=} \lim_{b\to\infty} \int_0^b xe^{-x}dx = \boxed{1 \quad \underline{\text{converges}}}$$

Integration by Parts: Let $\boxed{\begin{array}{ll} u = x & e^{-x}dx = dv \\ du = dx & -e^{-x} = v \end{array}}$

$$\int_0^b xe^{-x}dx = \int_0^b u\,dv = uv\Big|_0^b - \int_0^b v\,du$$

$$= -xe^{-x}\Big|_0^b + \int_0^b e^{-x}dx$$

$$= -be^{-b} + 0 + \frac{(e^{-x})}{(-1)}\Big|_0^b$$

$$= -be^{-b} - e^{-b} + e^0$$

$$= -(b+1)e^{-b} + 1$$

$$\lim_{b\to\infty} -(b+1)e^{-b} + 1 = \lim_{b\to\infty} \frac{-(b+1)}{e^b} + 1 \overset{\text{L.H.}}{=} \lim_{b\to\infty} \cancel{\frac{-1}{e^b}}^{\,0} + 1 = \boxed{1}$$

Looks like "$\frac{\infty}{\infty}$"

2. [10 points] Let

$$f(x,y) = \begin{cases} \frac{3x^2y^2}{x^3+y^6} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

a) Does the $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Consider the path $(x,y) = (y^2, y)$ (that is, along the path where $x = y^2$):

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(y^2,y)\to(0,0)} f(y^2,y) = \lim_{y\to 0} \frac{3(y^2)^2 y^2}{(y^2)^3 + y^6}$$

$$= \lim_{y\to 0} \frac{3\cancel{y^6}}{2\cancel{y^6}} = \boxed{\frac{3}{2}}$$

Consider the path $(x,y) = (0,y)$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} f(0,y) = \lim_{y\to 0} \frac{\overbrace{3(0)^2(\cancel{y^2})}^{\equiv 0}}{\cancel{0} + y^6} = \boxed{0}$$

The two limits are different, so the limit overall DNE.

b) Is $f(x,y)$ continuous at $(0,0)$?

$f(x,y)$ cannot be continuous since the limit DNE at $(0,0)$. Moreover $\lim_{y\to 0} f(y^2,y) = \frac{3}{2} \neq 0 = f(0,0)$.

3.[10 points] Solve the following first order separable initial value problem

$$\frac{dy}{dx} = (y-1)(y-2)$$

with $y(0) = 0$.

Rewrite as $\frac{1}{(y-1)(y-2)} \frac{dy}{dx} = 1$

Integrate: $\int \frac{dy}{(y-1)(y-2)} = \int dx \quad (*)$

Partial Fraction Decomposition: $\frac{1}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2}$

$$\Rightarrow \quad 1 = (y-1)(y-2)\left[\frac{A}{y-1} + \frac{B}{y-2}\right]$$

$$= (y-2)A + (y-1)B$$

$$= (A+B)y + (-2A-B)$$

$$\Rightarrow \begin{cases} A+B=0 \\ -2A-B=1 \end{cases}$$

$+$

$$-A + 0 = 1$$

$$\Rightarrow \boxed{A=-1} \Rightarrow \boxed{B=1} \Rightarrow \frac{1}{(y-1)(y-2)} = \frac{-1}{y-1} + \frac{1}{y-2}$$

$(*)$ $\int \frac{-1}{y-1} + \frac{1}{y-2}\, dy = \int dx \Rightarrow -\ln|y-1| + \ln|y-2| = x + C_1$

$$\Rightarrow \ln\left|\frac{y-2}{y-1}\right| = x + C_1 \Rightarrow \left|\frac{y-2}{y-1}\right| = e^{x+C_1} = C_2 e^x, \quad C_2 = e^{C_1}$$

$$\Rightarrow \quad \frac{y-2}{y-1} = Ce^{x}, \quad C = \pm C_2$$

$$\begin{array}{r|l} & \;\;1 \\ \cline{2-2} y-1 & y-2 \\ & -(y-1) \\ \cline{2-2} & \;\;-1 \end{array} \quad \Rightarrow \quad \frac{y-2}{y-1} = 1 - \frac{1}{y-1}$$

$$\Rightarrow \quad 1 - \frac{1}{y-1} = Ce^{x}$$

$$\Rightarrow \quad -\frac{1}{y-1} = Ce^{x} - 1$$

$$\Rightarrow \quad \frac{1}{y-1} = 1 - Ce^{x}$$

$$\Rightarrow \quad y - 1 = \frac{1}{1-Ce^{x}}$$

$$\Rightarrow \quad y = 1 + \frac{1}{1-Ce^{x}}$$

$$y(0) = 0 \quad \Rightarrow \quad 0 = 1 + \frac{1}{1-Ce^{0}} = 1 + \frac{1}{1-C}$$

$$\Rightarrow \quad -1 = \frac{1}{1-C}$$

$$\Rightarrow \quad 1 - C = -1$$

$$\Rightarrow \quad \boxed{C = 2}$$

$$\Rightarrow \quad \boxed{y = 1 + \frac{1}{1-2e^{x}}} \quad \text{or} \quad \boxed{y = \frac{2-2e^{x}}{1-2e^{x}}}$$ (combining fractions)

4. [10 points] Consider the system of linear equations

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & & & = & 0 \\ 3x_1 & + & x_2 & - & x_3 & = & 11 \\ 2x_1 & + & x_2 & + & 2x_3 & = & 11 \end{array}$$

Find the augmented matrix of the above system and use it to solve the system.

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 3 & 1 & -1 & 11 \\ 2 & 1 & 2 & 11 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 4 & 0 & -1 & 11 \\ 3 & 0 & 2 & 11 \end{array}\right] \xrightarrow{R_2 \to 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 8 & 0 & -2 & 22 \\ 3 & 0 & 2 & 11 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 11 & 0 & 0 & 33 \\ 3 & 0 & 2 & 11 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{11} R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 3 \\ 3 & 0 & 2 & 11 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 0 & -1 & 0 & -3 \\ 1 & 0 & 0 & 3 \\ 3 & 0 & 2 & 11 \end{array}\right] \xrightarrow{R_1 \to -R_1} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 3 & 0 & 2 & 11 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{2} R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} x_1 = 3 \\ x_2 = 3 \\ x_3 = 1 \end{array}}$$

6 pts 5. [15 points] Consider $f(x, y) = 3xy - x^3 - y^3$

a) Locate all critical points of $f(x, y)$.

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(3y - 3x^2, 3x - 3y^2\right)$$

critical pt. if $\nabla f(x,y) = (0,0) \Rightarrow \begin{cases} 3y - 3x^2 = 0 \\ 3x - 3y^2 = 0 \end{cases} \Rightarrow 3y = 3x^2$ or $y = x^2$

$\Rightarrow 3x - 3(x^2)^2 = 0$

$\Rightarrow 3x - 3x^4 = 0$

$\Rightarrow x(1 - x^3) = 0 \Rightarrow \boxed{x = 0}$ or $\boxed{x = 1}$

$x = 0 \Rightarrow y = 0^2 = 0$

Hence $\boxed{(0,0) \text{ is a critical pt.}}$

$x = 1 \Rightarrow y = 1^2 = 1$

Hence $\boxed{(1,1) \text{ is a critical pt.}}$

7 pts b) Classify the critical points of $f(x, y)$ (i.e determine if they are local max / local min or saddle point).

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -6x & 3 \\ 3 & -6y \end{pmatrix}$$

$$\det Hf(x,y) = (-6x)(-6y) - (3)(3) = 36xy - 9$$

- $\det Hf(0,0) = 36(0) - 9 = -9 < 0 \Rightarrow \boxed{(0,0) \text{ is a saddle pt.}}$
- $\left.\begin{array}{l} \det Hf(1,1) = 36(1) - 9 = 27 > 0 \\ \text{and } \frac{\partial^2 f}{\partial x^2}(1,1) = -6(1) = -6 < 0 \end{array}\right\} \Rightarrow \boxed{(1,1) \text{ a local max}}$

2 pts c) Does f have a global maximum or minimum on $\mathbb{R}^2$? Briefly explain!

$\boxed{\text{No.}}$ Let $y = 0$, we get $f(x,0) = -x^3$, whos range is all of $\mathbb{R}$, and so $f(x,y)$ has no absolute max or min.

6. [15 points] Consider the following system of differential equations

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

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a) Solve the following initial value problem with $x_1(0) = 5$ and $x_2(0) = 3$.

$A = \begin{bmatrix} -5 & -2 \\ 6 & 3 \end{bmatrix}$, $\det(A - \lambda I) = 0$

$\Rightarrow \det \begin{bmatrix} -5-\lambda & -2 \\ 6 & 3-\lambda \end{bmatrix} = (-5-\lambda)(3-\lambda) + 12 = 0$

$\Rightarrow -15 + 2\lambda + \lambda^2 + 12 = 0 \Rightarrow \lambda^2 + 2\lambda - 3 = 0$

$\Rightarrow (\lambda+3)(\lambda-1) = 0$

$\Rightarrow \boxed{\lambda_1 = -3}$ or $\boxed{\lambda_2 = 1}$ (eigenvalues)

- $\begin{bmatrix} -5 & -2 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = (-3)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \begin{cases} -5u_1 - 2u_2 = -3u_1 \\ 6u_1 + 3u_2 = -3u_2 \end{cases}$

$\boxed{\text{choose } \vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$ (eigenvector) $\Rightarrow -2u_2 = 2u_1 \Rightarrow \boxed{u_1 = -u_2}$

- $\begin{bmatrix} -5 & -2 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (1)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{cases} -5v_1 - 2v_2 = v_1 \\ 6v_1 + 3v_2 = v_2 \end{cases} \Rightarrow -2v_2 = 6v_1 \Rightarrow \boxed{v_2 = -3v_1}$

$\boxed{\text{choose } \vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}}$

(eigenvector)

$3 = 1 + 2$

b) Locate the equilibria of the system and determine their stabilities.

BACK

The only equilibria of this kind of a system is $\boxed{\text{the point } (0,0)}$ ✓

Since one of the eigenvalues is positive with the other negative

the the equilibria point is an $\boxed{\text{unstable saddle point}}$ ✓

Since $\lambda_1 \neq \lambda_2$, the general solution is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = c_1 e^{0} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{0} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 - 3c_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 = 5 \\ + \quad -c_1 - 3c_2 = 3 \end{cases}$$

$$0 \quad -2c_2 = 8$$

$$\Rightarrow \boxed{c_2 = -4} \Rightarrow c_1 - 4 = 5$$

$$\Rightarrow \boxed{c_1 = 9}$$

Hence the final (specific) solution is:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 9e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 4e^{t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9e^{-3t} - 4e^{t} \\ 12e^{t} - 9e^{-3t} \end{bmatrix}$$

<u>or</u>

$$\begin{cases} x_1(t) = 9e^{-3t} - 4e^{t} \\ x_2(t) = 12e^{t} - 9e^{-3t} \end{cases}$$

Therefore, the TOTAL number of different committees we can choose is $210+126+126$

$= \boxed{462}$ ✓

7. [10 points] A committee of 3 people are going to be chosen from a group of 9. Committee consists of a chairman, a secretary and a member. In how many ways we may select this committee if Paula and Cindy do not want to serve together.

We have 3 seats to fill, each of which is a different position. Therefore the order in which the seats (or positions) are assigned matters.

$\underset{C}{\_\_} \quad \underset{S}{\_\_} \quad \underset{M}{\_\_}$

Since there a 9 people, including Paula and Cindy, denote them by $P, C, G_1, \ldots, G_7$. That is there are 7 group members besides Paula and Cindy. Note that any permutation cannot include both Paula and Cindy.

- Suppose neither P nor C serve. Then the number of committees is

$\underset{C}{\underline{7}} \cdot \underset{S}{\underline{6}} \cdot \underset{M}{\underline{5}} =$ 210 different committees without P or C

- Suppose P serves. Then C does not, and there $7 \cdot 6$ ways to chose people from $G_1, \ldots, G_7$ to fill the two seats P does not. Since there are 3 positions P may hold, we have $3 \cdot (7 \cdot 6) =$ 126 different committees with P

- By the same argument we have 126 committes with C

8. [10 points] Let $f(x,y) = \sqrt{4x^2+y^2}$ be a function of two variables.

a) Compute the directional derivative of function $f(x,y)$ at the point $(-2,4)$ in the direction of $\vec{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

- $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{4x^2+y^2}\right) = \frac{\partial}{\partial x}(4x^2+y^2)^{1/2} = \frac{1}{2}(4x^2+y^2)^{-1/2} \cdot \frac{\partial}{\partial x}(4x^2+y^2)$

$$= \frac{1}{2}\frac{1}{\sqrt{4x^2+y^2}}(8x)$$

$$= \boxed{\frac{4x}{\sqrt{4x^2+y^2}}}$$

BACK $\longrightarrow$

- Similarly, $\frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{\sqrt{4x^2+y^2}}\frac{\partial}{\partial y}(4x^2+y^2) = \boxed{\frac{y}{\sqrt{4x^2+y^2}}}$

b) Find the angle between the vectors $\nabla f(-2,4)$ and $\vec{v}$.

$$\vec{x}\cdot\vec{y} = \|\vec{x}\|\,\|\vec{y}\|\cos\theta$$

$$\Rightarrow \nabla f(-2,4)\cdot\vec{v} = \|\nabla f(-2,4)\|\,\|\vec{v}\|\cos\theta$$

$$\Rightarrow (\nabla f(-2,4)\cdot\vec{v})\frac{1}{\|\vec{v}\|} = \|\nabla f(-2,4)\|\cos\theta$$

$$\Rightarrow \nabla f(-2,4)\cdot\vec{u} = \|\nabla f(-2,4)\|\cos\theta$$

$\sqrt{\frac{5}{2}}$

$$\Rightarrow \frac{\sqrt{5}}{2} = \sqrt{\left(\frac{-2}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\cos\theta \Rightarrow \frac{\sqrt{5}}{2} = \cancel{\sqrt{\frac{4}{2}+\frac{1}{2}}}\cos\theta \Rightarrow \cos\theta = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{4}}$$

$$\therefore \nabla f(-2,4) = \left(\frac{4(-2)}{\sqrt{4(-2)^2+(4)^2}} , \frac{4}{\sqrt{4(-2)^2+(4)^2}} \right)$$

$$= \left(\frac{-8}{\sqrt{2(16)}} , \frac{4}{\sqrt{2(16)}} \right) = \left(\frac{-8}{4\sqrt{2}} , \frac{4}{4\sqrt{2}} \right)$$

$$= \left(\frac{-2}{\sqrt{2}} , \frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}}(-2,1)$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{(-3)^2+(-1)^2}}(-3,-1) = \frac{1}{\sqrt{10}}(-3,-1)$$

$$\Rightarrow \quad D_{\vec{u}} f(-2,4) = \nabla f(-2,4) \cdot \vec{u}$$

$$= \frac{1}{\sqrt{2}}(-2,1) \cdot \frac{1}{\sqrt{10}}(-3,-1)$$

$$= \frac{1}{\sqrt{20}}\left((-2)(-3) + (1)(-1)\right)$$

$$= \frac{1}{\sqrt{20}}(5)$$

$$= \frac{1}{2\sqrt{5}}(5)$$

$$= \frac{5}{2\sqrt{5}}$$

$$= \boxed{\frac{\sqrt{5}}{2}} \checkmark$$

9. [10points] Assume that a car lot contains 20 percent Porches, 30 percent Volvos, and 50 percent BMWs. Of the Porches, 60 percent have two airbags, 10 percent of the Volvos have two airbags, and 30 percent of the BMWs have two airbags. You are assigned a car at random.

5 a) What is the probability that the car has two airbags?

Let A denote "has two airbags".

Let P denote "is a Porche", V "is a Volvo", B "is a BMW".

Then P, V, B form a <u>partition</u> of the car lot $\left(\Omega = P \cup V \cup B \text{ and } P \cap V = \emptyset,\ P \cap B = \emptyset,\ B \cap V = \emptyset \right)$

Then the total probability of A is given by:

$$\begin{aligned} Pr(A) &= Pr(A \cap P) + Pr(A \cap V) + Pr(A \cap B) \\ &= Pr(A|P)Pr(P) + Pr(A|V)Pr(V) + Pr(A|B)Pr(B) \\ &= (0.60)(0.20) + (0.10)(0.30) + (0.30)(0.50) = 0.30 \end{aligned}$$

or $\boxed{\text{30\% chance of getting a car with 2 airbags}}$

5 b) If the car has two airbags, what is the probability that it is a Porche?

We want to find $Pr(P|A)$:

Bayes' Formula: $$Pr(P|A) = \frac{Pr(A|P)Pr(P)}{Pr(A|P)Pr(P) + Pr(A|V)Pr(V) + Pr(A|B)Pr(B)}$$

$$= \frac{Pr(A|P)Pr(P)}{Pr(A)}$$

$$= \frac{(0.60)(0.20)}{(0.30)} = 0.4$$

or $\boxed{\text{40\% a car is a porche, if it has two airbags}}$