FINAL PRACTICE EXAM IV

YI LI

1. Let $f(x,y) = x^2 - y^2$ with constraint function

$$2x + y = 1.$$

Using Lagrange multipliers to find all extrema.

Write $g(x,y) = 2x + y - 1$. Then

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ -2y \end{bmatrix}, \quad \nabla g(x,y) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The equation $\nabla f(x,y) = \lambda \nabla g(x,y)$ implies

$$2x = 2\lambda, \quad -2y = \lambda, \quad 2x + y = 1.$$

thus $x = 2/3$ and $y = -1/3$, with $f(2/3, -1/3) = 1/3$.

2. Consider the system of linear equations

$$\begin{aligned} 2x - y + 3z &= 3 \\ 2x + y + 4z &= 4 \\ 2x - 3y + 2z &= 2 \end{aligned}$$

Find the augmented matrix of the above system and use it to solve the system.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 2 & 1 & 4 & 4 \\ 2 & -3 & 2 & 2 \end{array}\right]$$

Then

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 2 & 1 & 4 & 4 \\ 2 & -3 & 2 & 2 \end{array}\right] \xrightarrow[R_1 - R_3]{R_1 - R_2} \begin{array}{c} R_4 \\ R_5 \\ R_6 \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 0 & -2 & -1 & -1 \\ 0 & 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_5 + R_6} \begin{array}{c} R_7 \\ R_8 \\ R_9 \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $y = (1-z)/2$ and $x = 7(1-z)/4$. thus

$$(x,y,z) \in \left\{ \left(\frac{7}{4}(1-z), \frac{1}{2}(1-z), z \right) : z \in \mathbf{R} \right\}.$$

3. Let $f(x,y) = \sqrt{4 - x^2 - y^2}$.
(a) Find the largest possible domain and the corresponding range of $f(x,y)$.
(b) Compute $f_x(1,1)$ and $f_y(1,1)$.

(a) The domain is $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \leq 4\}$ and the range is $[0, \infty)$.
(b) $f_x(x,y) = -x(4 - x^2 - y^2)^{-1/2}$ and $f_y(x,y) = -y(4 - x^2 - y^2)^{-1/2}$. Hence

$$f_x(1,1) = \frac{-1}{\sqrt{2}} = f_y(1,1).$$

4. Compute

$$\int_0^1 \ln x \, dx.$$

Compute

$$\begin{aligned}\int_0^1 \ln x \, dx &= x \ln x \Big|_0^1 - \int_0^1 x \frac{1}{x} \, dx = x \ln x \Big|_0^1 - \int_0^1 dx \\ &= \left(0 - \lim_{x \to 0} x \ln x\right) - 1 = -1 - \lim_{x \to \infty} x \ln x = -1.\end{aligned}$$

5. Find the global extrema of

$$f(x,y) = x^2 - 3y + y^2, \quad -1 \leq x \leq 1, \ 0 \leq y \leq 2.$$

The function $f(x,y)$ has global extrema. Compute

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ -3 + 2y \end{bmatrix}, \quad \mathbf{Hess}(f)(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Letting $\nabla f(x,y) = \mathbf{0}$ we find that $(0, 3/2)$ is in the interior of the domain of f and $f(x,y)$ has a local minimum -2.25 at $(0, 3/2)$.

We now check the boundary values of $f(x,y)$.

(i) Consider the line segment $C_1 = \{(x,y) \in \mathbf{R}^2 : -1 \leq x \leq 1 \text{ and } y = 0\}$. On C_1, the function f is of the form

$$f(x,0) = x^2, \quad -1 \leq x \leq 1.$$

Hence the critical point of f on C_1 is $(0,0)$, and then f has global minimum 0 at $(0,0)$ and has the global maximum 1 at $(-1,0)$ and $(1,0)$, on the line segment C_1.

(ii) Consider the line segment $C_2 = \{(x,y) \in \mathbf{R}^2 : x = 1 \text{ and } 0 \leq y \leq 2\}$. On C_2, the function f is of the form

$$f(2,y) = 1 - 3y + y^2, \quad 0 \leq y \leq 2.$$

Hence, the critical point of f on C_2 is $(1, 3/2)$, and then f has the global minimum -1.25 and has the global maximum 1 at $(1,0)$, on the line segment C_2.

(iii) Consider the line segment $C_3 = \{(x,y) \in \mathbf{R}^2 : -1 \leq x \leq 1 \text{ and } y = 2\}$. On C_3, the function f is of the form

$$f(x,2) = x^2 - 2, \quad -1 \leq x \leq 1.$$

Hence, the critical point of f on C_3 is $(0,2)$, and then f has the global minimum -2 at $(0,0)$ and has the global maximum -1 and $(1,0)$ on the line segment C_3.

(iv) Consider the line segment $C_4 = \{(x, y) \in \mathbf{R}^2 : x = -1 \text{ and } 0 \leq y \leq 2\}$. On C_4, the function f is of the form

$$f(-1, y) = 1 - 3y + y^2, \quad 0 \leq y \leq 2.$$

Hence, the critical point of f on C_4 is $(-1, 3/2)$, and then f has the global minimum -1.25 at $(-1, 3/2)$ and has the global maximum 1 at $(-1, 0)$.

Therefore, the function has the global maximum 1 at $(-1, 0)$ and $(1, 0)$, and the global minimum -2.25 at $(0, 3/2)$.

6. Use the partial-fraction method to solve

$$\frac{dy}{dx} = (y-1)(y-2)$$

with $y(0) = 0$.

Compute

$$\frac{dy}{(y-1)(y-2)} = dx \Longrightarrow \left(\frac{1}{y-2} - \frac{1}{y-1}\right) dy = dx \Longrightarrow \ln\left|\frac{y-2}{y-1}\right| = x + C_1.$$

Thus

$$\frac{y-2}{y-1} = Ce^x \Longrightarrow y = \frac{2 - Ce^x}{1 - Ce^x}.$$

$y(0) = 0$ implies $C = 2$. Hence

$$y = \frac{2 - 2e^x}{1 - 2e^x}.$$

7. Find all candidates for local extrema and use the Hessian matrix to determine the type:

$$f(x, y) = e^{-x^2 - y^2}.$$

Compute

$$\nabla f(x, y) = \begin{bmatrix} -2x \\ -2y \end{bmatrix} e^{-x^2-y^2}.$$

The only critical point is $(0, 0)$. Since

$$\mathbf{Hess}(f)(x, y) = \begin{bmatrix} -2 + 4x^2 & 4xy \\ 4xy & -2 + 4y^2 \end{bmatrix} e^{-x^2-y^2} \Longrightarrow \mathbf{Hess}(f)(0, 0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

the function $f(x, y)$ has a local maximum at $(0, 0)$.

However, f is always nonpositive, $f(x, y)$ has the global maximum at $(0, 0)$.

8. Suppose that

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

(a) Compute $\det A$. Is A invertible?

(b) Suppose that

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Write $AX = B$ as a system of linear equations.

(c) Show that if

$$B = \begin{bmatrix} 3 \\ \frac{9}{2} \end{bmatrix}$$

then $AX = B$ has infinitely many solutions.

(a) $\det A = 0$. So A is not invertible.
(b) $2x + 4y = b_1$ and $3x + 6y = b_2$.
(c) Then the system in (b) reduces to one equation $2x + 4y = 3$. Then $AX = B$ has infinitely many solutions.

9. Solve the given initial-value problem

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

with $x_1(0) = 3$ and $x_2(0) = -1$.

Let

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

Since $\det A = -1$ and $\operatorname{tr} A = 0$, an eigenvalue λ satisfies $\lambda^2 - 1 = 0$. So $\lambda_1 = 1$ and $\lambda_2 = -1$.

For $\lambda_1 = 1$, we have

$$\mathbf{0} = (A - \lambda_1 I_2)\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Longrightarrow \mathbf{u} = u_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -1$, we have

$$\mathbf{0} = (A - \lambda_2 I_2)\mathbf{v} = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Longrightarrow \mathbf{v} = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence the general solution is

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

By the initial condition, we have

$$3c_1 + c_2 = 3, \quad 2c_1 = 4 \Longrightarrow c_1 = 2, \; c_2 = -3.$$

Thus $x_1(t) = 6e^t - 3e^{-t}$ and $x_2(t) = 2e^t - 3e^{-t}$.

10. Suppose that

$$\frac{dy}{dx} = y(4 - y).$$

(a) Find the equilibria of this differential equation.
(b) Compute the eigenvalues associated with each equilibrium and discuss the stability of the equilibria.

Write $g(y) = y(4 - y) = -y^2 + 4y$. Then $g'(y) = -2y + 4$.
(a) The equilibria are $y = 0$ and $y = 4$.
(b) Since $g'(0) = 4 > 0$ and $g'(4) = -4 < 0$, 0 is unstable and 4 is locally stable.

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