FINAL PRACTICE EXAM II

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1. Let $f(x,y) = x + y$ with constraint function

$$\frac{1}{x} + \frac{1}{y} = 1, \quad x \neq 0, \ y \neq 0.$$

Using Lagrange multipliers to find all local extrema. Are these global extrema?

Let $g(x,y) = \frac{1}{x} + \frac{1}{y} - 1$. From

$$\nabla f(x,y) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \nabla g(x,y) = \begin{bmatrix} \frac{-1}{x^2} \\ \frac{-1}{y^2} \end{bmatrix}$$

we have

$$1 = -\frac{\lambda}{x^2}, \quad 1 = -\frac{\lambda}{y^2}, \quad \frac{1}{x} + \frac{1}{y} = 1, \quad x \neq 0, \ y \neq 0.$$

Thus $y = x$ or $y = -x$. In the second case, we obtain $1 = \frac{1}{x} + \frac{1}{-x} = 0$, a contradiction. Hence $y = x$ and $1 = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}$. Consequently, $x = y = 2$. There is only one local extrema $(2,2)$ with $f(2,2) = 4$.

It is clear to see that this local extrema $(0,0$ is not global, since

$$\lim_{x \to 1\pm} f(x,y) = \lim_{x \to 1\pm} \left(x + \frac{x}{x-1} \right) = \lim_{x \to 1\pm} \frac{x^2}{x-1} = \pm\infty.$$

2. Consider the system of linear equations

$$\begin{aligned} -2x + 4y - z &= -1 \\ x + 7y + 2z &= -4 \\ 3x - 2y + 3z &= -3 \end{aligned}$$

Find the augmented matrix of the above system and use it to solve the system.

The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 4 & -1 & -1 \\ 1 & 7 & 2 & -4 \\ 3 & -2 & 3 & -3 \end{array}\right]$$

Then

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left[\begin{array}{ccc|c} -2 & 4 & -1 & -1 \\ 1 & 7 & 2 & -4 \\ 3 & -2 & 3 & -3 \end{array}\right] \xrightarrow[R_1 + \frac{2}{3}R_3]{R_1 + 2R_2} \begin{matrix} R_4 \\ R_5 \\ R_6 \end{matrix} \left[\begin{array}{ccc|c} -2 & 4 & -1 & -1 \\ 0 & 18 & 3 & -9 \\ 0 & \frac{8}{3} & 1 & -3 \end{array}\right]$$

$$\xrightarrow{R_5 - \frac{27}{4}R_6} \begin{matrix} R_7 \\ R_8 \\ R_9 \end{matrix} \left[\begin{array}{ccc|c} -2 & 4 & -1 & -1 \\ 0 & 18 & 3 & -9 \\ 0 & 0 & \frac{-15}{4} & \frac{45}{4} \end{array}\right]$$

Therefore, $z=-3$, $y=0$, and $x=2$.

3. Let

$$f(x,y)=\begin{cases}\frac{4xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0),\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

(a) Does the $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?
(b) Is $f(x,y)$ continuous at $(0,0)$?

(a) Along the line $c: y=mx$, we have

$$\lim_{(x,y)\to(0,0) \text{ along with } C} f(x,y)=\lim_{x\to 0}\frac{4mx^2}{x^2+m^2x^2}=\frac{4m}{1+m^2}.$$

Choosing different m yields different limits, we conclude that the limit does not exist.

(b) By part (a), f is discontinuous at $(0,0)$.

4. Determine whether

$$\int_{-\infty}^{\infty}\frac{1}{x^2-1}dx$$

is convergent.

Consider the function $f(x)=\frac{1}{x^2-1}$. This function becomes infinity at $x=\pm 1$. Write the improper integral as

$$\int_{-\infty}^{\infty}\frac{dx}{x^2-1}=\int_{-\infty}^{-1}\frac{dx}{x^2-1}+\int_{-1}^{1}\frac{dx}{x^2-1}+\int_{1}^{\infty}\frac{dx}{x^2-1}.$$

Since

$$\int\frac{dx}{x^2-1}=\int\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|,$$

it follows that

$$\lim_{x\to -1}\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|=+\infty,\quad \lim_{x\to 1}\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|=-\infty.$$

Hence the improper integral is divergent.

5. Find the absolute maxima and minima of $f(x,y)=x^2+y^2+x+2y$ on the disk $D=\{(x,y)\in\mathbf{R}^2: x^2+y^2\leq 4\}$.

The gradient is

$$\nabla f(x,y)=\begin{bmatrix}2x+1\\2y+2\end{bmatrix}$$

The critical point inside of D is $(-1/2,-1)$. The Hessian matrix of f is

$$\mathbf{Hess}(f)(x,y)=\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$$

which implies that f has a local minimum $f(-1/2,-1)=-5/4$ at the point $-(-1/2,-1)$.

We next consider the boundary of D. Let $g(x,y) = x^2 + y^2 - 4$. Then we should consider the extrema problem with constraint:

$$f(x,y) = x^2 + y^2 + x + 2y \quad \text{with } g(x,y) = 0.$$

From the equation $\nabla f(x,y) = \lambda \nabla g(x,y)$, we see that if $f(x,y)$ has an extremum at (x_0, y_0) then

$$2x + 1 = 2\lambda x, \quad 2y + 2 = 2\lambda y, \quad x^2 + y^2 = 4.$$

The first two equations gives us $x = 1/(2\lambda - 2)$ and $y = 2/(2\lambda - 2)$; substituting them into the third one, we arrive at

$$\frac{1}{(2\lambda-2)^2} + \frac{4}{(2\lambda-2)^2} = 4 \Longrightarrow \lambda = 1 + \frac{\sqrt{5}}{4}.$$

Hence $x = 1/\sqrt{5}$ and $y = 4/\sqrt{5}$, with $f(2/\sqrt{5}, 4/\sqrt{5}) = 4 + 2\sqrt{5}$.

The absolute maxima is $4 + 2\sqrt{5}$ and the absolute minima is $-5/4$.

6. Use the partial-fraction method to solve

$$\frac{dy}{dt} = \frac{1}{2}y^2 - 2y$$

with $y(0) = -3$.

Compute

$$\frac{1}{2}dt = \frac{dy}{y(y-4)} = \frac{1}{4}\left(\frac{1}{y-4} - \frac{1}{y}\right) dy.$$

Hence

$$\frac{1}{4}\ln\left|\frac{y-4}{y}\right| = \frac{1}{2}x + C_1 \Longrightarrow \frac{y-4}{y} = Ce^{2x}.$$

Since $y(0) = -3$, we get $C = 7/3$ and then $y = 4/(1 - \frac{7}{3}e^{2x})$.

7. Find and classify the critical points of

$$f(x,y) = x^3 - 4xy + y, \quad (x,y) \in \mathbf{R}^2.$$

Compute

$$\nabla f(x,y) = \begin{bmatrix} 3x^2 - 4y \\ -4x + 1 \end{bmatrix}, \quad \mathbf{Hess}(f)(x,y) = \begin{bmatrix} 6x & -4 \\ -4 & 0 \end{bmatrix}$$

The only critical point is $(1/4, 3/64)$. Since $\det \mathbf{Hess}(f)(x,y) = -16 < 0$ for any points (x,y), it follows that $(1/4, 3/64)$ is a saddle point.

8. Compute the directional derivative of $f(x,y) = ye^{x^2}$ at $(0,2)$ in the direction $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

The gradient of f is

$$\nabla f(x,y) = \begin{bmatrix} 2xye^{x^2} \\ e^{x^2} \end{bmatrix} \Longrightarrow \nabla f(0,2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The normalization of $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is given by

$$\mathbf{u} = \frac{1}{\left\| \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\|} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{17}} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Then $D_{\mathbf{u}} f(0,2) = \nabla f(0,2) \cdot \mathbf{u} = -1/\sqrt{17}$.

9. Consider the following system of differential equations

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(a) Show that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has the repeated eigenvalues $\lambda_1 = \lambda_2 = 1$.

(b) Show that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of A and that any vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ can be written as

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(c) Show that

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a solution of the above system that satisfies the initial condition $x_1(0) = c_1$ and $x_2(0) = c_2$.

(a) $\det A = 1$ and $\operatorname{tr} A = 2$. Hence

$$0 = \lambda^2 - 2\lambda + 1 \Longrightarrow \lambda_1 = \lambda_2 = 1.$$

(b) Since

$$(A - I_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 = (A - I_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we verify the first part. The second part is obvious.

(c) By (b), we can rewrite $\mathbf{x}(t)$ as

$$\mathbf{x}(t) = e^t \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

then $x_1(0) = c_1$ and $x_2(0) = c_2$. Since any linear combination of two solutions is also a solution, $x(t)$ satisfies the above system.

10. Suppose that

$$\frac{dy}{dx} = (4 - y)(5 - y).$$

(a) Find the equilibria of this differential equation.

(b) Compute the eigenvalues associated with each equilibrium and discuss the stability of the equilibria.

Let $g(y) = (4-y)(5-y) = y^2 - 9y + 20$. Then $g'(y) = 2y - 9$.

(a) Two equilibria are $y = 4$ and $y = 5$.

(b) Since $g'(4) = -1 < 0$ and $g'(5) = 1 > 0$, it follows that the equilibrium 4 is locally stable while 5 is unstable.

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