Exam 1 Practice Problems

Solutions:

Problem 1

(a) Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$ then $\int_e^\infty \frac{1}{x(\ln x)^5}\,dx = \int_1^\infty \frac{1}{u^5}\,du$

and $\ln e = 1$

$$\int_1^\infty \frac{1}{u^5}\,du = \lim_{R\to\infty} \int_1^R \frac{1}{u^5}\,du = \lim_{R\to\infty} \frac{u^{-4}}{-4}\Big|_1^R = \lim_{R\to\infty}\left(-\frac{1}{4R^4} + \frac{1}{4}\right)$$

$$= \frac{1}{4}.$$

So the integral converges to $\frac{1}{4}$.

(b) $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} \int_c^1 x^{-1/2}\,dx = \lim_{c\to 0^+} 2\sqrt{x}\Big|_c^1 = 2$$

The integral converges to 2.

(c) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{\cos x}}\,dx$ Let $u = \cos x$

$\frac{du}{dx} = \sin x$

$$\Rightarrow \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} \quad \text{so}$$

$$\lim_{c\to\frac{\pi}{2}^-} \int_0^c \frac{\sin x}{\sqrt{\cos x}}\,dx = \lim_{c\to\frac{\pi}{2}^-} 2\sqrt{\cos x}\Big|_0^c = -2$$

(d) (i) $\lim_{x \to \infty} \frac{\ln x}{x} \underset{\uparrow}{=} \lim_{x \to \infty} \frac{1/x}{1} = 0$

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(ii) By definition of limit (i) means that given $c > 0$

there is $x_0 > 0$ such that for $x \geq x_0$

$$\frac{\ln x}{x} \leq c \implies \ln x \leq cx.$$

(iii) Apply (ii) with $c = \frac{1}{2p}$ then

$$\ln x \leq \frac{1}{2p} x$$

$$p \ln x \leq \frac{x}{2} \quad \text{substract } x \text{ from both sides}$$

$$p \ln x - x \leq -\frac{x}{2}$$

$$p \ln x + \ln e^{-x} \leq \ln e^{-x/2} \quad (\ln e = 1)$$

$$\ln(x^p \cdot e^{-x}) \leq \ln e^{-x/2} \quad \text{exponentiate both sides}$$

$$x^p \cdot e^{-x} \leq e^{-x/2} \quad \text{for } x \text{ is sufficiently large.}$$

(iv) Now let $R > 0$ be sufficiently large as in part (iii)

Then

$$\int_0^\infty x^p \cdot e^{-x}\, dx = \int_0^R x^p \cdot e^{-x}\, dx + \int_R^\infty x^p \cdot e^{-x}\, dx$$

$$\leq \int_0^R x^p \cdot e^{-x}\, dx + \int_R^\infty e^{-x/2}\, dx$$

Thus,
$$\int_0^\infty x^p e^{-x}\,dx \le \underbrace{\int_0^R x^p \cdot e^{-x}\,dx}_{\text{is finite}} + \underbrace{\int_0^\infty e^{-x/2}\,dx}_{\text{is convergent}}$$

is finite — it is just a definite integral of a cont. func.

is convergent — see class notes for 7.4

thus, $\int_0^\infty x^p e^{-x}\,dx$ converges. Note that this argument does not say anything about its value.

Problem 2.

(a)
$$\frac{dy}{dx} = y(y-5)$$

$$\int \frac{dy}{y(y-5)} = \int dx = x + C$$

Write $\frac{1}{y(y-5)} = \frac{A}{y} + \frac{B}{y-5}$ $\Rightarrow A(y-5) + By = 1$

so that $\begin{cases} A + B = 0 \\ -5A = 1 \end{cases}$

$A = -1/5$, $B = 1/5$

$$\int \left(-\frac{1}{5y} + \frac{1}{5(y-5)}\right) dy = x + C$$

$$-\frac{1}{5}\left(\ln|y| - \ln|y-5|\right) = x + C \Rightarrow \ln\left|\frac{y}{y-5}\right| = -5x + C$$

Then $\frac{y}{y-5} = C_1 e^{-5x}$ Now use $y(0) = 6$ to see $C_1 = 6$

$$\frac{y}{y-5} = 6e^{-5x} \Rightarrow y = 6e^{-5x}y - 30e^{-5x}$$

$$y - 6e^{-5x}y = -30e^{-5x}$$

$$(1-6e^{-5x})y = -30e^{-5x}$$

$$y = \frac{-30e^{-5x}}{1-6e^{-5x}}.$$

(b)

$$\frac{dy}{dx} = y^2(x - \sin x)$$

$$\int \frac{dy}{y^2} = \int (x-\sin x)\,dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + \cos x + C$$

$$y = \frac{-1}{\frac{x^2}{2} + \cos x + C}$$ use $y(0) = 1/2$

$$\frac{1}{2} = \frac{-1}{C} \Rightarrow C = -2$$

Hence, $$y = \frac{-1}{\frac{x^2}{2} + \cos x - 2}.$$

3. Let $g(y) = y(y^2-4)(3-y)$

(a) The equilibrium solutions are given by

$$g(y)=0 \iff y=0 \quad y=2,\ y=-2,\ y=3$$

To classify stability we need to use the stability criterion:

$$g'(y) = (y^2-4)(3-y) + 2y \cdot y(3-y) + y(y^2-4)(-1)$$

$$= -y^3 + 3y^2 + 4y - 12 + 6y^2 - 2y^3 - y^3 + 4y$$

$$= -4y^3 + 9y^2 + 8y - 12$$

$g'(0) = -12 < 0 \implies y=0$ is stable.

$g'(2) > 0 \implies y=2$ is unstable

$g'(-2) > 0 \implies y=-2$ is unstable

$g'(3) < 0 \implies y=3$ is stable.

(b) Since y is a solution and $y(0) = \frac{5}{2} \in [2,3]$ and there is no other equilibrium solution between 2 and 3, 2 is unstable 3 is stable we must have $\lim_{x\to\infty} y(x) = 3$.

4. (a) The system has a unique solution if

$$\frac{a}{3} \neq \frac{2}{a-1}$$

i.e. $a^2 - a - 6 \neq 0$

$(a-3)(a+2) \neq 0$

$a \neq 3$ and $a \neq -2$. <u>Ans:</u> all $a \in \mathbb{R} - \{-2, 3\}$

(b) The system has infinitely many solutions if

$$\underbrace{\frac{a}{3} = \frac{2}{a-1}} = \frac{4}{-1}$$

By above computation $a = 3$ or $a = -2$ in each case

<u>Ans:</u>

$\frac{a}{3} \neq \frac{4}{-1}$ Hence there is <u>NO</u> such a

(c) If $a = 3$ OR $a = -2$ then

$\frac{a}{3} = \frac{2}{a-1} \neq \frac{4}{-1}$ Hence no solution

<u>Ans.</u> $a = 3$ OR $a = -2$.

Problem 5 : The augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & -1 & 2 & 1 \\ 0 & 3 & -1 & -2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 3 & -1 & -2 \\ 0 & 3 & -1 & -2 \end{bmatrix} \begin{matrix} R_4 = R_1 \\ R_5 = R_1 - R_2 \\ R_6 = R_3 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_7 = R_4 \\ R_8 = R_5 \\ R_9 = R_5 - R_6 \end{matrix}$$

We parametrize the solution : $z = t$

$$R_8 \longrightarrow 3y - z = -2$$
$$3y - t = -2$$
$$3y = t - 2$$
$$\boxed{y = \frac{1}{3}t - \frac{2}{3}}$$

$$R_7 \longrightarrow x + 2y + z = -1$$
$$x + \frac{2}{3}t - \frac{4}{3} + t = -1$$
$$x = -\frac{5}{3}t + \frac{1}{3}$$

So the solution set

$$S = \begin{cases} x = -\frac{5}{3}t + \frac{1}{3} \\ y = \frac{1}{3}t - \frac{2}{3} \\ z = t \end{cases} \quad t \in \mathbb{R}$$

Problem 6 :

I.

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 1 \\ 1 & -1 & 0 & 0 \\ 3 & 1 & -1 & 2 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \longrightarrow \left[\begin{array}{cccc} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & -2 & -7 & -1 \end{array}\right]\begin{array}{l} R_4 = R_1 \\ R_5 = R_1 - R_2 \\ R_6 = R_3 - 3R_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{cccc} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right]\begin{array}{l} R_7 = R_4 \\ R_8 = R_5 \\ R_9 = R_5 + R_6 \end{array}$$

$$R_9 \longrightarrow -5z = 0$$

$$\boxed{z = 0}$$

$$R_8 \longrightarrow 2y + 2.z = 1$$

$$2y = 1$$

$$\boxed{y = 1/2}$$

$$R_7 \longrightarrow x + y + 2z = 1$$

$$x + \frac{1}{2} = 1$$

$$\boxed{x = 1/2}$$

Solution set = $\{(1/2, 1/2, 0)\}$.

II.

$$\left[\begin{array}{cccc} -1 & 0 & 2 & 1 \\ 2 & 1 & -3 & 5 \\ 1 & 1 & -1 & -3 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \longrightarrow \left[\begin{array}{cccc} -1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 7 \\ 0 & 1 & 1 & -2 \end{array}\right]\begin{array}{l} R_4 = R_1 \\ R_5 = 2R_1 + R_2 \\ R_6 = R_1 + R_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{cccc} -1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & -9 \end{array}\right]\begin{array}{l} R_7 = R_4 \\ R_8 = R_5 \\ R_9 = R_6 - R_5 \end{array}$$

$$R_9 \rightarrow 0.x + 0y + 0.z = -9$$

$$0 = -9 \ ?$$

No solution !

Problem 7:

(a) $\det(A) = 1.3 - 5.3$

$= -12$

(b) $A^{-1} = -\frac{1}{12}\begin{bmatrix} 3 & -5 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1/4 & 5/12 \\ 1/4 & -1/12 \end{bmatrix}$

(c) $\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 5 \\ 3 & 3-\lambda \end{bmatrix}\right)$

$= (1-\lambda)(3-\lambda) - 5.3$

$= \lambda^2 - 4\lambda + 3 - 15$

$= \lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2)$

$\lambda = 6$ and $\lambda = -2$ are the eigenvalues of A.

Eigenvectors

(1) Associated to $\lambda = 6$ $\quad Ax = 6x$

$\begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 6\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{cases} x_1 + 5x_2 = 6x_1 \\ 3x_1 + 3x_2 = 6x_2 \end{cases}$

$\Rightarrow \begin{cases} x_1 - x_2 = 0 \\ x_1 - x_2 = 0 \end{cases}$

Ans. All vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1 - x_2 = 0$

$\boxed{15}$ Eg. $x_1 = 1$ $x_2 = 1$ $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Associated to $\lambda = -2$ $\qquad Ax = -2x$

$$\begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} x_1 + 5x_2 = -2x_1 \\ 3x_1 + 3x_2 = -2x_2 \end{cases} \Rightarrow \begin{matrix} 3x_1 + 5x_2 = 0 \\ 3x_1 + 5x_2 = 0 \end{matrix}$$

All vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of the form $3x_1 + 5x_2 = 0$

E.g. $x_1 = 5,\ x_2 = -3$ $\qquad \vec{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

Problem 8:

(a) The vector equation of the line

$$\vec{n} \cdot (r - r_0) = 0$$

$$\begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - (-2) \end{bmatrix} = 0$$

$$2(x-1) + 3(y+2) = 0$$

$$2x + 3y + 4 = 0$$

(b) Let $x = t$ then $3y = -2t + 4$

$$y = -\frac{2}{3}t + \frac{4}{3}$$

$$\begin{cases} x = t \\ y = -\frac{2}{3}t + \frac{4}{3} \end{cases} \quad t \in \mathbb{R}.$$