FINAL EXAM: CALC II (BIO AND SOC. SCI.)

Problem 1. **a)** Integrate $dy = (3t-1)dt$: $y = \int(3t-1)dt = 3t^2/2 - t + C$. Determine the constant: $y(2) = 6 - 4 + C = 5 \Rightarrow C = 1$. So $y(t) = 3t^2/2 - t + 1$.
b) Integrate $\frac{dy}{1-y} = 2dt$: $-\ln(1-y) = 2t + C_1$. Exponentiate: $1 - y = Ce^{-2t}$, so $y(t) = 1 - Ce^{-2t}$. Determine the constant: $y(0) = 1 - C = 2 \Rightarrow C = -1$. So $y(t) = 1 + e^{-2t}$.
c) This is the autonomous equation $\frac{dy}{dt} = g(y)$ with $g(y) = y^5$. Equilibrium solution: $\hat{y} = 0$. The first derivative criterion is inconclusive since $g'(\hat{y}) = g'(0) = 0$. However g changes sign near $\hat{y}$ and since $g(y) > 0$ for $y > \hat{y}$, this means that $\hat{y} = 0$ is an unstable equilibrium.

Problem 2. **a)** $\det A = 2$ and $\det B = 0$, so A is invertible and B is not.
$A^{-1} = \frac{1}{2}\begin{pmatrix} -3 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -3/2 & -1/2 \\ -1/2 & -1/2 \end{pmatrix}$.
b) $AX = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \Rightarrow X = A^{-1}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{pmatrix} -3/2 & -1/2 \\ -1/2 & -1/2 \end{pmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$.

Problem 3. **a)** $\overrightarrow{QR} = \begin{bmatrix} -1 \\ -7 \end{bmatrix} \Rightarrow |\overrightarrow{QR}| = \sqrt{1^2 + (-7)^2} = \sqrt{50}$.
b) $\cos P = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|}$. But $\overrightarrow{PQ} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$. Hence $\overrightarrow{PQ}\cdot\overrightarrow{PR} = 0$. Therefore $\cos P = 0 \Rightarrow P = \frac{\pi}{2} = 90^\circ$.
c) The equation of the line passing through $(1,2)$ and perpendicular on the vector $\overrightarrow{QR} = \begin{bmatrix} -1 \\ -7 \end{bmatrix}$ is $(-1)(x-1) + (-7)(y-2) = 0 \Rightarrow x + 7y - 15 = 0$.

Problem 4. **a)** $f_x = \sin(\pi xy) + \pi xy\cos(\pi xy)$. $f_y = \pi x^2\cos(\pi xy)$.
b) $\nabla g = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}$ so $\nabla g(-1,2) = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$. Normalize direction: $\overrightarrow{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
$D_{\overrightarrow{u}}g(-1,2) = \nabla g(-1,2)\cdot u = \begin{bmatrix} -4 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1/\sqrt{2} \\ 1\sqrt{2} \end{bmatrix} = \frac{-3}{\sqrt{2}}$.
c) We need a unit vector normal to the gradient. Rotate the gradient vector by 90°: $R_{90}\nabla g(-1,2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$. Normalize: $\mathbf{v} = \frac{1}{17}\begin{bmatrix} -1 \\ -4 \end{bmatrix}$.
Alternative answer: $\frac{1}{17}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Problem 5. **a)** $f_x = 2x - 1 = 0 \Rightarrow x = 1/2$ and $f_y = 2y + 2 = 0 \Rightarrow y = -1$, so $(1/2,-1)$ is the only critical points. The Hessian $H_f(1/2,-1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has both eigenvalues positive, so it's positive definite. So $(1/2,-1)$ is a local minimum.
b) The interior of D contains no critical points, so the global maximum and minimum are achieved on the boundary. The values of f on the boundary are: $f(x,0) = x^2 - x$, $f(x,1) = x^2 - x + 3$, $f(0,y) = y^2 + 2y$, $f(1,y) = y^2 + 2y$. The

Date: May 3, 2007.

function $x^2 - x$ has a critical point at $x = 1/2$, while $y^2 + 2y$ has no critical points inside the interval $(0,1)$. Hence we have to compare the values of f at the following points: $(0,0), (0,1), (1,0), (1,1), (1/2,0)$ and $(1/2,1)$. We see that $\max f = 3$ is achieved at $(0,1)$ and $(1,1)$, while $\min f = -1/4$ is achieved at $(1/2,0)$.

Problem 6. Let X, Y denote the value of the first and second dice outcome.
a) The positive outcomes are: $44, 45, 46, 54, 55, 56, 64, 65, 66$. Total possible outcomes: 36. So $P(X, Y \geq 4) = \frac{9}{36} = \frac{1}{4} = 25\%$.
b) $P(X \geq 5 | S \geq 8) = \dfrac{P(X \geq 5 \& S \geq 8)}{P(S \geq 8)}$.
Compute $P(X \geq 5 \& S \geq 8)$. Positive outcomes: $53, 54, 55, 56, 62, 63, 64, 65, 66$. Hence $P(X \geq 5 \& S \geq 8) = \frac{9}{36}$.
Find $P(S \geq 8)$. Pos. outcomes: $26, 35, 36, 44, 45, 46, 53, 54, 55, 56, 62, 63, 64, 65, 66$. Hence $P(S \geq 8) = \frac{15}{36}$. Therefore $P(X \geq 5 | S \geq 8) = \frac{9/36}{15/36} = \frac{9}{15} = 60\%$.
c) $E[S] = E[X+Y] = E[X] + E[Y] = 2E[X]$. $\mathrm{var}(S) = \mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y) = 2\,\mathrm{var}(X)$ (independence).
X takes possible values $1, 2, 3, 4, 5, 6$ with equal odds: $1/6$. Therefore

$$E[X] = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$$
$$E[X^2] = \frac{1^2 + 2^2 + \cdots + 6^2}{6} = \frac{91}{6}$$
$$\mathrm{var}(X) = E[X^2] - (EX)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Hence $E[S] = 7$ and $\mathrm{var}(S) = \frac{35}{6}$.

Problem 7. a) $E[X+Y] = E[X] + E[Y] = 40 + 50 = 90$. By independence: $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y) = 6^2 + 8^2 = 100$, hence $\sigma(X+Y) = 10$.
b) $P(X \geq 43) = P(\frac{X-40}{6} \geq \frac{43-40}{6}) = P(Z \geq 0.5) = 1 - P(X \leq 0.5) = 0.3083$. $P(X \geq 34) = P(Z \geq -1) = P(Z \leq 1) = 0.84$.
c). The normal distribution of mean 0 and standard deviation $\sigma = \frac{1}{\sqrt{2}}$ has probability density function $f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$. Therefore $\int \frac{1}{\sqrt{\pi}} e^{-x^2} dx = 1$, hence $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$.

Problem 8. a) $P(S_6 = 4) = \frac{1}{2^6} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \dfrac{1}{64} \dfrac{6 \cdot 5}{1 \cdot 2} = \dfrac{15}{64}$.
b) We need to find $P\left(\left|\dfrac{X_1 + \cdots + X_{400}}{400} - \mu\right| \leq \dfrac{\mu}{100}\right) = P(|S_{400} - 400\mu| \leq 4\mu)$.
Since $E[S_{400}] = 400\mu$ and $\mathrm{var}(S_{400}) = 400\sigma^2 = 400$, by the Central Limit Theorem, $\dfrac{S_{400} - 400\mu}{20} = Z \sim N(0,1)$. Therefore

$$P(|S_{400} - 400\mu| \leq 4\mu) = P\left(\left(\left|\frac{S_{400} - 400\mu}{20}\right|\right) \leq \frac{\mu}{5}\right.$$
$$= P(|Z| \leq 1) = 2P(Z \leq 1) - 1 = 2 \cdot 0.84 - 1 = 0.68 = 68\%$$