1) (10 points) Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

a) Compute the inverse of A.

Solution: The inverse is just

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

b) Find the matrix X satisfying the equation $AXA + A = I$ (where I is the identity matrix).

Solution: By simple algebra the equation is equivalent to $XA+I = A^{-1}$, which means that $XA = A^{-1} - I$, or $X = A^{-1}A^{-1} - A^{-1}$. Thus,

$$\begin{aligned} X &= \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & -1/4 \end{pmatrix} \end{aligned}$$

2) (10 points) Suppose that

$$\frac{dy}{dx} = y(1-y)(y-2).$$

a) Find the equilibria of this differential equation.

Solution: The equilibria are $\hat{y} = 0$, $\hat{y} = 1$ and $\hat{y} = 2$.

b) Which of the equilibria are stable?

Solution: The right hand side of the equation is $g(y) = -y(y-1)(y-2)$. Since $g'(0) = -2$, $\hat{y} = 0$ is stable. Since $g'(1) = 1$, $\hat{y} = 1$ is unstable. Since $g'(2) = -2$, $\hat{y} = 2$ is stable.

3) (10 points) a) Genes relating to albinism are denoted by A and a. Only those people who receive gene a from both parents are albino. Persons having the gene pair (A, a) are normal in appearance and, because they can pass the trait to their offspring, they are called CARRIERS. Suppose that a normal in appearance couple has two children exactly one of whom is albino. Suppose that the nonalbino child mates with a person who is known to be a carrier for albinism. What is the probability that their first child is albino?

Solution: We first calculate the probability that the nonalbino child will pass on the a gene. This child is the offspring of two Aa parents. The sample space of offspring for these parents is $\{(A,A),(a,a),(A,a),(a,A)\}$. Since we know that the child is not of type (a,a), there is a $2/3$ chance that the child is of gene type Aa and a one third chance that it is of type AA. Thus, the chance of this child passing on the a gene must be $\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{3}$. On the other hand, the other mate is known to be a nonalbino carrier, i.e., of type Aa and the probability that this person passes down the a gene is $1/2$. Therefore, the probability that the first offspring will be albino is $\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}$, since the two events are independent.

b) A study was conducted among a certain group of union members whose health insurance policies required second opinions prior to surgery. Of those members whose doctors advised them to have surgery, 20% were informed by a second doctor that no surgery was needed. Of these, 70% took the second doctor's advice and did not go through with the surgery. Of the members who were advised to have surgery by both doctors, 95% went through with the surgery. What is the probability that a union member who had surgery was advised to do so by a second doctor?

Solution: Let B_1 be the event that surgery was recommended by the second doctor, and B_2 be the event that no surgery was recommended by the second doctor. Let A be the event that surgery was performed. The problem is asking what is $P(B_2|A)$. By Bayes formula,

$$P(B_2|A)=\frac{P(A|B_2)P(B_2)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)}.$$

Note that $P(A|B_2)=0.95$, and $P(B_2)=0.8$. Also, $P(A|B_1)=0.3$, and $P(B_1)=0.2$. Thus, the desired probability is

$$P(B_2|A)=\frac{(0.95)(0.8)}{(0.3)(0.2)+(0.95)(0.8)}.$$

4) (10 points) Find the tangent plane of the graph of $f(x,y)=\sin(xy)$ at the point $(1,0,0)$.

Solution: Note that

$$\frac{\partial f}{\partial x}=y\cos(xy),\quad \frac{\partial f}{\partial y}=x\cos(xy).$$

Thus,

$$\frac{\partial f(1,0)}{\partial x}=0,\quad \frac{\partial f(1,0)}{\partial y}=1.$$

Therefore, the equation of the tangent plane is $z - 0 = 0(x-1) + 1(y-0)$, i.e.,

$$z = y.$$

5) (10 points) a) A student is to answer 7 out of 10 questions in an examination. How many choices does she have? (Assume the order in which she answers the questions is unimportant.)

Solution: It's just the combination $\binom{10}{7}$.

b) A student is to answer 7 out of 10 questions in an examination. How many choices does he have if he must answer at least three of the first five questions? (Assume that the order in which he answers the questions is unimportant.)

Solution: It's

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2},$$

since three terms are the ways of answering 3 out of the first 5 questions, 4 out of the first 5, and 5 out of the first 5, respectively, given that 7 questions are answered.

6) (5 points) Explain why

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

Solution: Let X_k be the random variable that takes on values 0 and 1 with probability 1/2. Let $S_n = \sum_{k=0}^{n} X_k$. Then S_n is a random variable which takes on values $k = 0, 1, \ldots, n$. Its probabilities are given by the binomial distribution, $P(S_n = k) = \binom{n}{k}(1/2)^k(1/2)^{n-k} = 2^{-n}\binom{n}{k}$. Since $1 = \sum_{k=0}^{n} P(S_n = k)$, we have

$$1 = \sum_{k=0}^{n} \binom{n}{k} 2^{-n},$$

which leads to the formula after multiplying both sides of this equation by 2^n.

7) (25 points) Consider the linear system of two differential equations

$$\frac{dx_1(t)}{dt} = 10x_1(t) - 6x_2(t)$$
$$\frac{dx_2(t)}{dt} = 2x_1(t) + 2x_2(t).$$

a) Write the system in matrix form

$$\frac{dX(t)}{dt} = AX(t).$$

Solution: If $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, and

$$A = \begin{pmatrix} 10 & -6 \\ 2 & 2 \end{pmatrix},$$

then the equation takes this form.

b) Find the eigenvalues λ_i of the matrix A you wrote in part a).

Solution: The eigenvalue equation is

$$0 = \det \begin{pmatrix} 10-\lambda & -6 \\ 2 & 2-\lambda \end{pmatrix} = \lambda^2 - 12\lambda + 32 = (\lambda - 8)(\lambda - 4),$$

and so the eigenvalues are $\lambda_1 = 8$ and $\lambda_2 = 4$.

c) For each eigenvalue λ_i of the matrix A above find an eigenvector u_i corresponding to the eigenvalue λ_i.

Solution: An eigenvector for λ_1 is a vector $\begin{pmatrix} v \\ w \end{pmatrix}$ solving

$$\begin{pmatrix} 10 & -6 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 8 \begin{pmatrix} v \\ w \end{pmatrix},$$

which is equivalent to $2v - 6w = 0$. Thus,

$$u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda_1 = 8$.

Eigenvectors for $\lambda_2 = 4$ must solve

$$\begin{pmatrix} 10 & -6 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 4 \begin{pmatrix} v \\ w \end{pmatrix},$$

which is equivalent to $6v - 6w = 0$, and so

$$u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda_2 = 4$.

d) Write the general solution $X(t)$ of the linear system of differential equations.

Solution: $X(t) = c_1 e^{8t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where c_1 and c_2 are constants.

e) Now assume the initial condition $x_1(0) = -1$ and $x_2(0) = -2$ is given. Find the solution of this initial value problem.

Solution: By the initial conditions

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix} = X(0) = \begin{pmatrix} 3c_1 + c_2 \\ c_1 + c_2 \end{pmatrix},$$

which means the constants must solve the system

$$\begin{aligned} 3c_1 + c_2 &= -1 \\ c_1 + c_2 &= -2 \end{aligned}$$

Subtracting the second equation from the first leads to $2c_1 = 1$, and so $c_1 = 1/2$, which in turn means that $c_2 = -2 - c_1 = -5/2$. Thus, the solution of this initial value problem is

$$e^{8t} \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} - e^{4t} \begin{pmatrix} 5/2 \\ 5/2 \end{pmatrix}.$$

8) (10 points) Recall that a random variable U is said to be uniformly distributed over the interval $(2, 3)$ if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{3-2}, & 2 \leq x \leq 3 \\ 0, & \textit{otherwise}. \end{cases}$$

a) Compute $P(1.5 \leq U \leq 2.5)$.

Solution: It equals $P(2 \leq U \leq 2.5) = 0.5$.

b) Find the distribution function $F(x)$ of the random variable U.

Solution: $F(x) = \int_{-\infty}^{x} f(w)dw$. Thus, F vanishes on $(-\infty, 2)$, while it equals $\int_2^x 1 \cdot dw = x - 2$ on $(2, 3)$, and 1 when $x \geq 3$:

$$F(x) = \begin{cases} 0, & x \leq 2 \\ x - 2, & 2 < x < 3 \\ 1, & x \geq 3. \end{cases}$$

c) Compute EU (the expected value of U).

Solution: $EU = \int_{-\infty}^{\infty} x f(x)dx = \int_2^3 x dx = \frac{1}{2} \cdot (3^2 - 2^2) = 5/2$.

d) Compute the variance of U.

Solution: We first compute $EU^2 = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_2^3 x^2 dx = \frac{1}{3} \cdot (3^3 - 2^3) = 19/3$. Thus,

$$\text{var}(X) = EU^2 - (EU)^2 = \frac{19}{3} - \frac{25}{4} = \frac{1}{12}.$$

e) Compute the standard deviation of U.

Solution: It's just the square root of the variance: $\sqrt{1/12}$.

9) (10 points) Sam's Club customers have weight (measured in pounds) which is normally distributed with mean 225 and standard deviation 25. Find the probability that at least one of the five random Sam's Club shoppers in line in front of you weighs more than 275 pounds.

Solution: Let X be the random variable which is the weight of a random Sam's Club customer. If Z is the standard normal, and $F(z)$ its distribution function. Then the probability that a random Sam's club customer weighs less than 275 pounds is

$$P(X \leq 275) = F(Z \leq \frac{275 - 225}{25}) = F(Z \leq 2) = 0.9772.$$

Therefore, the probability that all 5 customers in line in front of you weigh less than 275 is $(0.9772)^5$, which means that the probability that at least one of the customers weighs more than 275 is $1 - (0.9772)^5$.