1) Denote by $L(t)$ the length of a fish at time t, and assume that the fish grows according to the von Bertalanffy equation

$$\frac{dL}{dt} = k(34 - L(t)), \quad \text{with } L(0) = 2,$$

where k is a constant. Solve this equation.

Solution: After separating variables we get $\frac{dL}{34-L} = k\,dt$. If we integrate both sides we get $-\ln|34 - L| = kt + C_1$, and so

$$L - 34 = (\pm e^{-C_1})e^{-kt} = Ce^{-kt}.$$

Thus, $L(t) = 34 + Ce^{-kt}$. Since $2 = L(0) = 34 + C$, we get that $C = -32$ from the initial condition, which means that the solution of the equation is

$$L(t) = 34 - 32e^{-kt}.$$

2) Suppose that a deer population evolves according to the logistic equation, and that a fixed number of deer per unit time are removed due to hunting. That is,

$$\frac{dN}{dt} = rN\big(1 - \frac{N}{K}\big) - H.$$

Assume that $r = 1$, $K = 100$ and $H = 24$. Find all possible equilibria and discuss their stability.

Solution: The equation is $\frac{dN}{dt} = g(N) = N(1 - \frac{N}{100}) - 24$. The equilibria will be the $N = \hat{N}$ for which $g(\hat{N}) = 0$. That is they will be the roots

$$0 = N(1 - \frac{N}{100}) - 24 = -\frac{N^2}{100} + N - 24.$$

Using the quadratic formula we see that the two roots are

$$\frac{-1 \pm \sqrt{1 - \frac{96}{100}}}{-\frac{2}{100}},$$

which simplifies to 50 ± 10. Thus, the two equilibria are $\hat{N} = 60, 40$.

To discuss their stability, we note that

$$g'(N) = (1 - \frac{N}{100}) - \frac{N}{100} = 1 - \frac{2N}{100}.$$

Since $g'(40) = 1 - 80/100 = 1/5 > 0$ and $g(60) = 1 - 120/100 = -1/5 < 0$, we conclude that $\hat{N} = 40$ is unstable, and $N = 60$ is stable.

3) Suppose that A is a 2×5 matrix B is a 1×3 matrix, C a 5×1 matrix and D a 2×3 matrix. Which of the following matrix multiplications are defined? Whenever it is defined, state the size of the resulting matrix.

(a) AB, **Solution:** It is not defined.

(b) AC, **Solution:** It is defined and is a 2×1 matrix.

(c) BD, **Solution:** It is not defined.

(d) BC, **Solution:** It is not defined.

4) For which constant c is the function

$$f(x) = \begin{cases} ce^{-4x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

a density function? Show your work.

Solution:

$$\int_0^\infty e^{-4x} dx = \lim_{N \to \infty} \int_0^N e^{-4x} dx = \lim_{N \to \infty} -\frac{1}{4}(e^{-4N} - 1) = \frac{1}{4},$$

and therefore, one must have $c = 4$ for $\int_{-\infty}^{\infty} f(x)dx = 1$, which is the condition that f is a density.

5) If X is distributed according to $f(x)$ in the previous problem, find the mean of X.

Solution: To compute the mean of X we note that

$$\int_0^N 4xe^{-4x} dx = -xe^{-4x}]_0^N + \int_0^N e^{-4x} dx.$$

The first term on the right is $-Ne^{-4N}$ which tends to zero as $N \to \infty$, while by 4) the integral tends to $1/4$. Thus, the mean of X is $\int_0^\infty 4xe^{-4x} dx = \frac{1}{4}$.

6) Find the inverse (if it exists) of

$$C = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}.$$

If the inverse does not exist, explain why.

Solution: The determinant of C is $12 - 12 = 0$. Since matrices with zero determinant are not invertible, we conclude that the inverse of C does not exist.