Math 107: Calculus II, Spring 2006: Midterm Exam II Solutions

1. *Calculate each of the following partial derivatives:*

   (a) if $f(x,y) = \cos(xy)$*, find* $\dfrac{\partial^2 f}{\partial x\,\partial y}$*;*

   (b) if $f(x,y) = x^2y + x^3y^3 + 1$*, find* $\dfrac{\partial^2 f}{\partial x^2}$*;*

   (c) if $f(x,y) = (x+y)^{-1/2}$*, find* $\dfrac{\partial^2 f}{\partial x\,\partial y}$*.*

   (a) $\dfrac{\partial f}{\partial y} = -x\sin(xy)$ and so $\dfrac{\partial^2 f}{\partial x\,\partial y} = -\sin xy - xy\cos(xy)$ (by the product rule);

   (b) $\dfrac{\partial f}{\partial x} = 2xy + 3x^2y^3$ and so $\dfrac{\partial^2 f}{\partial x^2} = 2y + 6xy^3$;

   (c) $\dfrac{\partial f}{\partial y} = -\dfrac{1}{2}(x+y)^{-3/2}$ and so $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{3}{4}(x+y)^{-5/2}$.

2. *Let f be the function of two variables given by*

$$f(x,y) = xy(x-y)$$

   (a) Calculate the gradient vector ∇f and evaluate at the point $(2,1)$.

   (b) What is the directional derivative of f at $(2,1)$ in the direction of the vector $(1,1)$?

   (c) In what direction is the directional derivative of f at $(2,1)$ the least (i.e. the most negative)?

   (a) Writing $f(x,y) = x^2y - xy^2$ we get $\nabla f = (2xy - y^2, x^2 - 2xy)$. At the point $(2,1)$ this is $(3,0)$.

   (b) We can use the formula

$$D_{(1,1)}f(2,1) = \frac{\nabla f(2,1)\cdot(1,1)}{|(1,1)|} = \frac{(3,0)\cdot(1,1)}{\sqrt{2}} = 3/\sqrt{2}.$$

   Alternatively, we can make $(1,1)$ into a unit vector by dividing by its length. This gives $(1/\sqrt{2}, 1/\sqrt{2})$. Then we have

$$D_{(1,1)}f(2,1) = \frac{\partial f}{\partial x}(2,1)\frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y}(2,1)\frac{1}{\sqrt{2}} = 3/\sqrt{2} + 0 = 3/\sqrt{2}.$$

   (c) The directional derivative is least in the opposite direction to the gradient vector, that is, in the direction of $-\nabla f = (-3,0)$.

3. *Find the linear approximation to the function $f(x,y) = e^{2x}\cos 3y$ at the point $(0,0)$. Use your approximation to get an estimate of the value of $f(0.1, 0.1)$.*

   The formula for the linear approximation to f at the point (a,b) is

   $$f(x,y) \simeq f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

   In our case we have

   - $f(0,0) = e^0 \cos 0 = 1$
   - $\dfrac{\partial f}{\partial x} = 2e^{2x}\cos 3y$, so $\dfrac{\partial f}{\partial x}(0,0) = 2$
   - $\dfrac{\partial f}{\partial y} = -3e^{2x}\sin 3y$, so $\dfrac{\partial f}{\partial x}(0,0) = 0$

   Therefore the linear approximation is:

   $$f(x,y) \simeq 1 + 2(x-0) + 0(y-0) = 1 + 2x.$$

   Therefore

   $$f(0.1, 0.1) \simeq 1 + 2 \times 0.1 = 1.2.$$

4. *Find the critical point of the function $f(x,y) = e^{xy}$. Is this critical point a local max, a local min, or a saddle? (Show your work.)*

   To find the critical point, we set the partial derivatives equal to zero. That is:

   $$\frac{\partial f}{\partial x} = ye^{xy} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = xe^{xy} = 0.$$

   Since e^{xy} is never zero, these mean that $x = 0$ and $y = 0$. So $(0,0)$ is the only critical point. To decide if it is a local max, a local min or a saddle, we have to find the Hessian matrix. The second-order partial derivatives are:

   - $\dfrac{\partial^2 f}{\partial x^2} = y^2 e^{xy}$
   - $\dfrac{\partial^2 f}{\partial x\,\partial y} = e^{xy} + xye^{xy}$ (using the product rule)
   - $\dfrac{\partial^2 f}{\partial y^2} = x^2 e^{xy}$

   Therefore the Hessian is given by

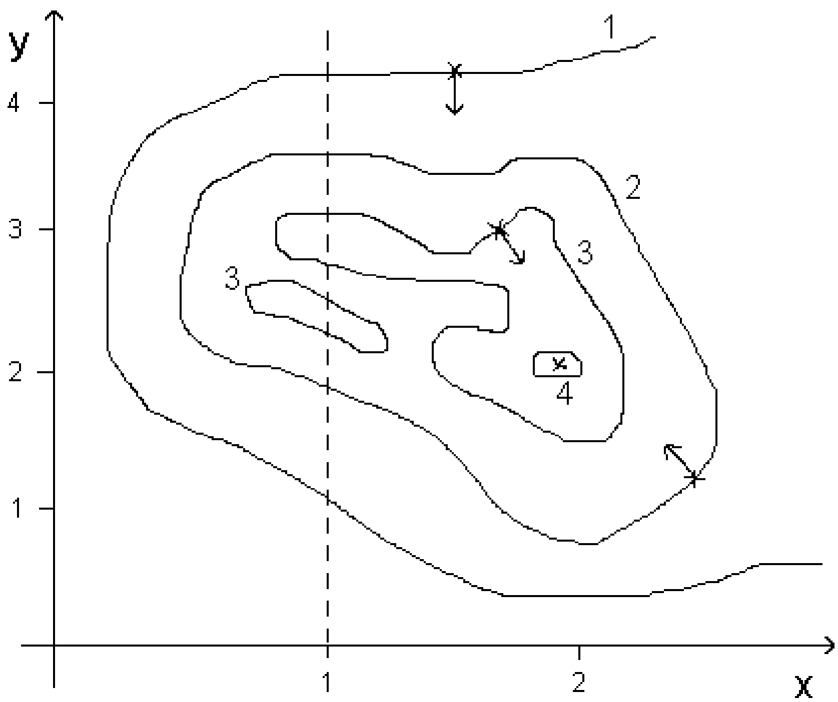
   $$Hf = \begin{pmatrix} y^2 e^{xy} & e^{xy} + xye^{xy} \\ e^{xy} + xye^{xy} & x^2 e^{xy} \end{pmatrix}.$$

   Evaluating at the critical point $(0,0)$ we get

   $$Hf(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   This has determinant $0 \times 0 - 1 \times 1 = -1$. This is negative and so the critical point $(0,0)$ is a saddle.

5. *The following diagram displays some of the level curves of a function $f(x, y)$ of two variables. The number labelling a curve 1, 2, 3 or 4 denotes the value of the function f along that curve.*

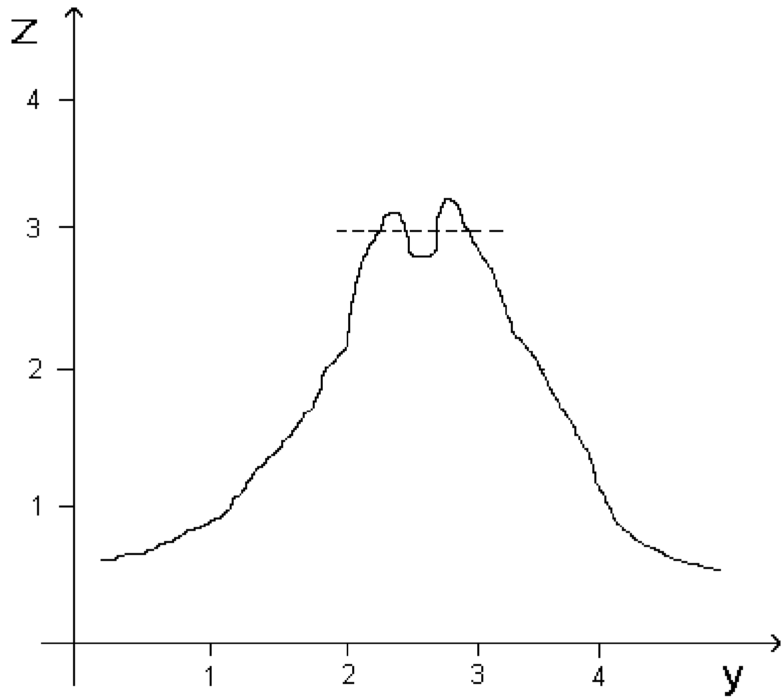


(a) *Sketch the $x = 1$ cross-section through the graph of the function f. (Your graph should be a 2-dimensional graph of z against y, with z giving the value of the function at a particular y. Label the axes of your graph as fully as possible.)*

(b) *At each of the three points marked with a cross, draw an arrow that represents the direction of the gradient vector for f at that point. (You should draw the arrows directly on the above diagram.)*

(c) *Based on the information in the picture, at roughly what point (x, y) would you expect the global maximum of the function f to be?*

(a) As y increases from 0 up to 4, the value of the function (and hence the z-coordinate in the cross-section) first increases, passing the 1, 2 and 3 level curves. But after crossing the 3-level curve once, it crosses it again. This means that the function is dipping back below 3. But then it crosses back over the 3-level curve, so is going back above 3, before decreasing past the 3, 2 and 1 level curves. The graph below displays this cross-section, showing in particular that the function crosses the 3-level curve (i.e. the dotted line $z = 3$ four times altogether. You needed to draw this graph fairly accurately in order to get full credit for this problem.



(b) The gradient vectors are marked on the diagram above. In each case, they are perpendicular to the level curve, and pointing in the direction of increasing f, that is, towards the next higher level curve.

(c) The largest the function gets seems to be slightly above 4, and the global maximum probably occurs inside the 4-level curve, at around the point $(2, 2)$ (marked with a cross). Notice that this function probably also has another *local* maximum inside the smaller 3-level curve, at around $(0.9, 2.5)$.