110.107 CALCULUS II FINAL EXAMINATION - 14 December 2001

Instructions: Please read these instructions. No calculators or books are allowed. Please do your scratch work on the blank sheets provided or on the backs of the test sheets. When you are satisfied with your work, write your answers legibly and coherently on the question sheets in the space provided. Justify your answers completely. When in doubt, provide more rather than less explanation. **A table for the standard normal distribution is appended.**

1. Let $L(t)$ denote the length of a fish at time t and assume that the fish grows according to the von Bertalanffy equation
$$\frac{dL}{dt} = k(40 - L(t)) \text{ with } L(0) = 4$$
If $L(5) = 22$, find the value of the constant k and find the asymptotic length of the fish.

2. Suppose that $N(t)$ denotes the size of a population at time t and that
$$\frac{dN}{dt} = 1.5N\left(1 - \frac{N}{100}\right)$$
Solve this differential equation when $N(0) = 50$ and determine the size of the population in the long run (i.e., calculate $\lim_{t\to\infty} N(t)$).

3. Let $p = p(t)$ be the fraction of occupied patches in a metapopulation model, and assume that
$$\frac{dp}{dt} = 2p(1-p) - p \text{ for } t \geq 0$$
Find all equilibria of the above equation that are in the interval [0,1] and determine their stability.

4. Suppose that a drug is administered to a person in a single dose, and assume that the drug does not accumulate in body tissue but is excreted through urine. Let $x_1(t)$ be the concentration in the body after t hours and $x_2(t)$ the concentration in the urine after t hours. If the original dose is 100mg and it takes 30 minutes for the drug to be at one-half of its initial concentration in the body, find a system of differential equations for x_1 and x_2.

5. A simple two-compartment model for gap dynamics in a forest assumes that gaps are created by disturbances and that they revert back to forest as the trees grow in the gaps. Let $x_1(t)$ denote the area occupied by the gaps, and $x_2(t)$ the area occupied by the adult trees. If the dynamics are given by the system
$$\begin{aligned}\frac{dx_1}{dt} &= -0.3x_1 + 0.2x_2\\ \frac{dx_2}{dt} &= 0.3x_1 - 0.2x_2\end{aligned}$$

compute the eigenvalues and eigenvectors for this system, and give the general solution.

6. Chemotaxis is the chemically directed movement of organisms up a concentration gradient. The slime mold *Dictyostelium discoideum* exhibits this phenomenon; single-celled amoeba of this species move up the concentration gradient of a chemical called cyclic AMP. If the concentration of the cyclic AMP at the point (x,y) in the xy-plane is given by

$$f(x, y) = 6\sqrt{2xy + 1}$$

determine in which direction an amoeba at the point (3,4) will move if its movement is directed by chemotaxis.

7. Suppose the mass of a certain animal is normally distributed with a mean of 3750 g and a standard deviation of 350 g. What percentage of the population has a mass between 3225 g and 3925 g?

8. Consider the exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

a) Show that the mean of this distribution is $1/\lambda$.

b) Recall that if $f(x)$ is a continuous distribution, then the *median* is defined as the number m such that

$$\int_{-\infty}^{m} f(x)\,dx = \int_{m}^{\infty} f(x)\,dx = \frac{1}{2}$$

Calculate the median of the exponential distribution above.

9. Suppose that the number of seeds that a plant produces is normally distributed with a mean of 200 and a standard deviation of 25. Find the probability that in a sample of 4 plants, at least one produces more than 250 seeds.