Math 107: Calculus II, Fall 2006: Practice Final

The following questions should give you a feel for the style of the final exam. It is approximately the same length and contains the same length/difficulty/types of questions as the exam. You should try to make sure you can do all these problems without your notes. The following instructions will appear on the exam

1. There are ten questions. Each is worth 20 points.
2. **Do not open your booklet until told to begin.** The exam will be 3 hours long.
3. You may **not** use calculators, books, notes or any other paper. Write all your answers on this booklet. Additional paper is available if required.
4. **Show all your working!** In some questions, points are available for giving reasons for your answers. Unless you are particularly told you only need to give the answer, you should show all your working and explain what you are doing.
5. **READ THE QUESTIONS CAREFULLY!** Do the questions you can do first. Then come back to the ones that look hard.

1. Consider the autonomous differential equation
$$\frac{dy}{dt} = y^3.$$
   (a) Find the equilibrium solution and sketch the phase line for this differential equation.
   (b) Draw a sketch of the solutions to this equation. (You should make your sketch based on the phase line you drew in part (a). You do not need to solve the equation.)
   (c) Is the equilibrium solution stable, unstable or semi-stable?
   (d) Find the general solution to this differential equation. (Make sure your answer includes the equilibrium solution.)
   (e) Find the specific solution with the initial condition $y(0) = 1/4$.
2. Find the general solution to the following system of differential equations:
$$\frac{dx}{dt} = -2x + y; \quad \frac{dy}{dt} = y.$$
   Give your answer in the form $x(t) = \ldots$, $y(t) = \ldots$.
3. (a) For what values of a does the following matrix not have an inverse?
$$\begin{pmatrix} a & 4 \\ -1 & 2 \end{pmatrix}$$

(b) Solve the following system of linear equations:

$$\begin{pmatrix} a & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

in each of the following cases:

i. $a = 1$;

ii. $a = -2$.

4. (a) Let $f(x, y) = e^{x+y-xy}$. Find the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$.

   (b) Find the gradient vector ∇f at the point $(1, 0)$.

   (c) Find the directional derivative of f at the point $(1, 0)$ in the direction of the vector $(1, 1)$.

   (d) Find the linear approximation to f at the point $(1, 0)$.

5. This question concerns the function $f(x, y) = y + x^2$.

   (a) Sketch the c-level curves of the function f for $c = -2, 0, 2$. (Put all your curves on one graph, making sure to label each curve with its value of c.)

   (b) Draw the $y = 1$ cross-section of the function f. Draw the tangent line to this graph whose slope represents the partial derivative

$$\frac{\partial f}{\partial x}(1, 1).$$

   (c) Calculate the gradient vector ∇f at the point $(1, 1)$ and mark this on your diagram from part (a).

6. (a) Show that $(0, 0)$, $(4, 0)$ and $(2, 2)$ are critical points of the function

$$f(x, y) = x^2y - 4xy + y^2.$$

   (b) Classify these critical points as local maxima, local minima or saddle points.

7. My sock drawer contains two red socks, two white socks and two blue socks. Each morning I pick socks out of the drawer one by one until I have a pair that match.

   (a) Find the probabilities of each of the following outcomes of this random experiment:

      i. RBWR

      ii. RR

   (b) Find the probabilities of each of the following events:

      i. the first three socks I pick out are all different colors

      ii. the first two socks I pick out are the same color

   (c) Find the expectation of the random variable X = total number of socks.

8. The continuous random variable X has cumulative distribution function

$$F(x) = P(X \leq x) = \begin{cases} 0 & \text{for } x \leq 0; \\ x(2-x) & \text{for } 0 \leq x \leq 1; \\ 1 & \text{for } x \geq 1. \end{cases}$$

   (a) Find the probability density function $f(x)$ of the random variable X. Sketch a graph of $f(x)$.

   (b) Calculate $P(0.5 \leq X \leq 1)$.

   (c) Find the expectation and variance of X.

9. A long-distance runner's times for the marathon are normally distributed with mean 2.3 hours and standard deviation 0.1.

   (a) Find the probability that in a given race, the runner breaks the world record of 2 hours and 6 minutes (or 2.1 hours).

   (b) The runner runs 16 races one year. Write down an expression for the probability that the runner breaks the world record at some point during the year. (You do not need to evaluate the expression, but you should put it in a form that could easily be entered into a calculator to find the answer.)

   (c) Use the Central Limit Theorem to approximate the probability that the runner's average time over the 16 races is less than 2.225 hours.

   (Useful information: if Z is a standard normal distribution, then $P(Z \leq 2) = 0.9772$ and $P(Z \leq 3) = 0.9986$.)

10. An experiment was performed to find the expectation of a random variable X. Data from 100 observations had sample mean 10.24 and sample standard deviation 1.2.

   (a) Use this data to find a 95.44% confidence interval for the expectation EX. (You can use the fact that a 95.44% confidence interval is of the form $\hat{\mu} \pm 2 \times S.E.$)

   (b) An experiment to observe a different random variable Y was also performed with sample mean 1.5 and sample standard deviation 1. A further experiment to observe the random variable $X + Y$ had sample mean 11.7 and sample variance 5.2. Based on all the information in this question, would you expect that the random variables X and Y are independent or not? (Warning: note carefully when you are given a variance and when you are given a standard deviation.)