Math 107: Calculus II, Spring 2006: Midterm I Solutions

1. (a) *Show that*
$$y = (1-3x)^{-1/3}$$
is a solution to the differential equation
$$\frac{dy}{dx} = y^4.$$
(b) *Find the equilibrium solution and sketch the phase line for this differential equation.*
(c) *Classify the equilibrium as stable, unstable or semi-stable.*

(a) To show that a particular function is a solution to a differential equation, we substitute it in. Taking $y = (1-3x)^{-1/3}$ we get
$$\frac{dy}{dx} = -\frac{1}{3}(1-3x)^{-4/3}(-3) = (1-3x)^{-4/3}$$
and
$$y^4 = ((1-3x)^{-1/3})^4 = (1-3x)^{-4/3}.$$
These are equal and so the given function is indeed a solution.

(b) This question was done very badly by almost everyone. You are asked to find the equilibrium solution for the differential equation. The equilibrium is where $dy/dx = 0$. We get
$$y^4 = 0$$
and so the equilibrium is at $y = 0$.
Many, many people set $y = 0$ in the function $y = (1-3x)^{-1/3}$ and solved to find $x = 1/3$. This does not make sense. The equilibrium is a constant solution to the equation and so must be $y =$ something. [Incidentally, it is not even true that putting $x = 1/3$ gives you $(1-3x)^{-1/3} = 0$. In fact, we have
$$(1-3x)^{-1/3} = \frac{1}{\sqrt[3]{1-3x}}$$
and putting $x = 1/3$ gives
$$\frac{1}{\sqrt[3]{0}} = \frac{1}{0}$$
which is not defined.]
The phase line is a picture of the y-axis. There is an dot marking the constant solution at $y = 0$. For $y > 0$, y^4 is positive, so we have an arrow showing solutions are increasing. For $y < 0$, y^4 is also positive, so the solutions there are also increasing.

(c) This is a semi-stable equilibrium.

2. *Find the general solution to the separable equation*

$$\frac{dy}{dt} = y^2 e^t.$$

We solve this by separating the variables and integrating to get:

$$\int \frac{1}{y^2}\,dy = \int e^t\,dt.$$

This works out as

$$\frac{-1}{y} = e^t + c$$

(where c is the constant of integration). Then we re-arrange to find y:

$$y = \frac{-1}{e^t + c}.$$

We must also find the 'missing solutions' that we lose when we divide by y^2. This occur when $y^2 = 0$, that is $y = 0$. Therefore the general solution is:

$$y = \frac{-1}{e^t + c} \quad \text{or} \quad y = 0.$$

3. (a) *Does the matrix* $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ *have an inverse? Give a reason for your answer.*

   (b) *For each of the following matrix equations, say if there are no solutions, exactly one solution, or infinitely many solutions:*

   i. $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$;

   ii. $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

   (a) To decide if a matrix has an inverse, we calculate its determinant:

   $$\det\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = 2 \times 2 - 1 \times 4 = 0.$$

   Therefore the matrix is singular and does not have an inverse.

   (b) i. Writing the matrix equation as two simultaneous equations we get

   $$\begin{aligned} 2x + 4y &= -6 \\ x + 2y &= -3 \end{aligned}$$

   If we multiply the second equation by two, we get the same as the first equation. This means that we can take any value of y and set $x = -2y - 3$ to get a solution. So there are infinitely many solutions.

ii. In this case , the equations are

$$\begin{aligned} 2x + 4y &= 5 \\ x + 2y &= 3 \end{aligned}$$

Now if we multiply the second equation by 2, we get $2x + 4y = 6$ which is inconsistent with the first equation. Therefore, there are no solutions.

4. (a) *The matrix* $\begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$ *has only one eigenvalue. What is it?*

   (b) *Find an eigenvector corresponding to your eigenvalue from part (a).*

   (c) *Use your answers to parts (a) and (b) to write down a solution to the following system of differential equations:*

$$\frac{dx}{dt} = 4x - y; \quad \frac{dy}{dt} = x + 2y.$$

(Note: in part (c), you do ***not*** *have to find the general solution - we didn't cover finding the general solution when there is only one eigenvalue. You should write down the specific solution given by the eigenvalue and eigenvector that you found in parts (a) and (b).)*

(a) To find the eigenvalue, we find the determinant of the matrix

$$\begin{pmatrix} 4-k & -1 \\ 1 & 2-k \end{pmatrix}.$$

This is

$$(4-k)(2-k) - (1)(-1) = k^2 - 6k + 8 + 1 = k^2 - 6k + 9 = (k-3)^2.$$

The eigenvalue is the value of k that makes this zero, so it is $k = 3$.

(b) To find an eigenvector, we have to solve

$$\begin{pmatrix} 4-k & -1 \\ 1 & 2-k \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Setting $k = 3$ and writing as equations we get

$$\begin{aligned} u - v &= 0 \\ u - v &= 0 \end{aligned}$$

Therefore $u = v$. Since we can choose any u, v that satisfy these equations, we can take the eigenvector to be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (But any multiple of this would also have worked.)

(c) Using our eigenvalue and eigenvector, a solution to the given system of equations is

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or, equivalently,

$$x(t) = e^{3t}, \quad y(t) = e^{3t}.$$

5. *I am looking to invest \$1000 that I recently won in the lottery. My three main options pay interest on my investment via the following differential equations (where $M(t)$ is the amount of my investment (measured in dollars) at time t):*

- *Bank A:* $\dfrac{dM}{dt} = (M - 1000)^3$
- *Bank B:* $\dfrac{dM}{dt} = (M - 100)$
- *Bank C:* $\dfrac{dM}{dt} = (M - 1000000)^3$

Which bank should I invest in to maximize the return from my \$1000? Explain your answer.

The best way to answer this question is to draw sketches of the solutions to each differential equation, and then compare what happens in each case if I start with \$1000. Bank A has an equilibrium at 1000, is increasing above 1000 and decreasing below. Therefore, if I start with 1000, I will always stay at 1000. (Some people argued that if my bank balance gets even a little bit about 1000, then it will increase really fast. This is true, but on the other hand, if it gets below 1000 then it will go to zero. In any case, the question said you start with \$1000 so in Bank A that would not change.)

Bank B has an equilibrium at \$100, is increasing above and is decreasing below. Investing \$1000 here would be good, because it would increase (exponentially) to infinity. (In fact, you could have solved the differential equation and got the function $M(t) = 100 + 900e^t$ which describes how your money would grow.)

Bank C has an equilibrium at \$1,000,000, is increasing above and is decreasing below. If you start with \$1000 here, it will decrease to zero.

On the basis of these observations, Bank B is the one that maximizes your return from investing \$1000.