Math 107, Spring 2006: Midterm II Practice Exam Solutions

Solutions

1. *This question concerns the function*

$$f(x, y) = 1 - yx^2.$$

   (a) *Draw the cross-section through the graph of $f(x, y)$ given by setting $y = 1$.*

   (b) *Calculate the partial derivative of f with respect to x and evaluate it at the point $(2, 1)$.*

   (c) *On your graph from (a), draw a tangent line to the cross-section whose slope is represented by your answer from (b).*

   (a) The graph should be of z against x with $z = 1 - x^2$ (from setting $y = 1$).

   (b) $\dfrac{\partial f}{\partial x} = -2xy$, which at the point $(2, 1)$ is -4.

   (c) Part (b) tells us that the slope of the graph at $x = 2$ should be -4.

2. (a) *Draw a graph showing the c-level curves of the function $f(x, y) = x^2 - y$ for $c = -2, 0, 2$.*

   (b) *Mark on your graph from (a) the direction of the gradient vector ∇f at the points $(0, 0)$, $(2, 6)$ and $(-1, -1)$. (Note: you do not have to get the right length for these gradient vectors. It is the direction that is important.)*

   (c) *What is the directional derivative of $f(x, y)$ in the direction $(1, 0)$ at the point $(0, 0)$? (You must give a reason or a calculation.)*

   (a) The c-level curve is the curve with equation $x^2 - y = c$, or $y = x^2 - c$. So the level curves are parabolas.

   (b) The gradient vectors should be perpendicular to the level curves, and should point in the direction of increasing f.

   (c) The directional derivative can be calculated as

$$D_{\mathbf{u}} f(0, 0) = \frac{\partial f}{\partial x}(0, 0)u_1 + \frac{\partial f}{\partial y}(0, 0)u_2.$$

   The partial derivatives are:

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -1$$

   So the directional derivative is:

$$0 \times 1 + (-1) \times 0 = 0.$$

   Alternatively, you could just say that the vector $(1, 0)$ is perpendicular to the gradient vector at $(0, 0)$, and so the directional derivative in that direction is zero.

3. *Find the critical point of the function $f(x, y) = x^2 + 3y^2$. Is this a local max, local min or saddle? (Show your work.)*

   The critical point occurs where both partial derivatives are zero. So, we need $2x = 0$ and $6y = 0$. Therefore, $x = 0$ and $y = 0$. So the critical point is $(0, 0)$.

   To see if this is a local max/min or saddle, we calculate the Hessian matrix:

$$Hf = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}.$$

   This has determinant $+12$, so $(0, 0)$ is either a local max or min. We then check the top-left and bottom-right entries. These are both positive, so the critical point is a local minimum.

4. *Calculate each of the following partial derivatives:*

   (a) if $f(x, y) = e^{x^2 y}$, find $\dfrac{\partial^2 f}{\partial y^2}$;

   (b) if $f(x, y) = \sin(2x + y)$, find $\dfrac{\partial^2 f}{\partial x \partial y}$;

   (a) $\dfrac{\partial f}{\partial y} = x^2 e^{x^2 y}$ and so $\dfrac{\partial^2 f}{\partial y^2} = x^4 e^{x^2 y}$.

   (b) $\dfrac{\partial f}{\partial y} = \cos(2x + y)$ and so $\dfrac{\partial^2 f}{\partial x \partial y} = -2\sin(2x + y)$.

5. *Use the chain rule to find $f'(t)$ where $f(x, y) = x^2 y$, $x(t) = \sin t$, $y(t) = e^t$. (Your answer should be in terms of t only.)*

   The chain rule says that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

   In our case: $\partial f/\partial x = 2xy$, $dx/dt = \cos t$, $\partial f/\partial y = x^2$ and $dy/dt = e^t$. Therefore

$$f'(t) = 2xy \cos t + x^2 e^t.$$

   Substituting in the expressions for x and y, we get:

$$f'(t) = 2e^t \sin t \cos t + e^t (\sin t)^2.$$