Math 107, Spring 2006: Midterm I Practice Exam

The following questions are meant to give you an idea of the length and difficulty of the questions on the exam on Tuesday. Bear in mind that topics that do not appear here can, and probably will, be on the real thing.

The following set of instructions will appear on the exam on Tuesday:

1. There are five questions. Each is worth 20 points.
2. **Do not open your booklet until told to begin.** The exam will be 50 minutes long.
3. You may **not** use calculators, books, notes or any other paper. Write all your answers on this booklet. Additional paper is available if required.
4. **You must show all your working to obtain full credit!**
5. **Read the questions carefully!** Some questions only require an answer, others require particular explanations. If in doubt, write more!

Questions

1. (a) Find the general solution to the differential equation

$$\frac{dy}{dt} = 3t^2 + e^t.$$

   (b) Find the specific solution to this equation that has the initial condition

$$y(0) = 3.$$

2. Consider the autonomous differential equation

$$\frac{dy}{dt} = g(y)$$

   where $g(y) = (1 - y)y$.

   (a) Find the equilibrium solutions of this equation.

   (b) Use the first derivative test to classify the stability of these equilibria.

   (c) Sketch the solutions to this differential equation.

3. (a) Find the determinant of the matrix $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$.

   (b) Find the inverse of the matrix from part (a).

(c) Use your answer to (b) to solve the system of linear equations

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}.$$

4. (a) Show that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$, and find the corresponding eigenvalues.

(b) Find the general solution to the following system of differential equations (write your answer in the form $x(t) = \dots, y(t) = \dots$):

$$\frac{dx}{dt} = 2x; \quad \frac{dy}{dt} = x + 3y.$$

5. A colony of penguins has $P(t)$ members at time t (measured in years). The colony's size varies as a result of reproduction and penguins dying. If there are fewer than 10 members, they will all die off. If there are more than 10 members, the size of the colony will increase, up to a maximum of 100. At either 10 or 100 members, the colony is in equilibrium.

(a) The colony's size satisfies a differential equation

$$\frac{dP}{dt} = g(P)$$

for some function $g(P)$. Draw the graph of a possible function $g(P)$ based on the information given about how the size of the colony varies.

(b) Are the equilibria stable or unstable?