Math 107: Calculus II, Spring 2008: Practice Final III

1. (a) Find the solution to the pure time differential equation

$$\frac{dy}{dt} = 3t^2 + 2t$$

that satisfies the condition $y(0) = 2$.

(b) An autonomous differential equation has a stable equilibrium at $y = 1$ and an unstable equilibrium at $y = 4$. Draw the phase line for this equation and sketch possible solution curves.

(c) Draw the graph of a function $g(y)$ such that the differential equation

$$\frac{dy}{dt} = g(y)$$

would have the behaviour described in part (b).

2. (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

(b) Use your answers to part (a) to write down the general solution to the system of differential equations

$$\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = -2x + 4y.$$

(c) The system of equations from part (b) has an equilibrium at $(0, 0)$. Is this equilibrium stable or unstable?

3. (a) Find the inverse of the matrix

$$\begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$$

(b) Solve the following matrix equation

$$\begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

(c) Find values for the constants a, b such that the matrix equation

$$\begin{pmatrix} 0 & a \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 2 \end{pmatrix}$$

has

i. exactly one solution;

   ii. infinitely many solutions;
   iii. no solutions.

   (Note: the three cases are separate – you will find a different pair of values for a, b in each case.)

4. This question concerns the function

$$f(x, y) = x^2 + xy + y.$$

   (a) Find the partial derivatives of f.
   (b) Find the linear approximation to f at the point $(1, 2)$.
   (c) Find the directional derivative of f at $(1, 2)$ in the direction $(-1, 1)$.

5. This question concerns the function

$$f(x, y) = 4 - x^2 - y^2.$$

   (a) Sketch the c-level curves of f for $c = -5, 0, 3$. (Put all your curves on the same diagram. Make sure you label each curve with the corresponding value of c.)
   (b) Calculate the gradient vector ∇f.
   (c) Mark on your picture from part (a) the direction of the gradient vector at the points $(0, 1)$, $(2, 0)$ and $(0, -3)$. (Label these three points on your diagram and make sure each appears on the correct level curve.)
   (d) In which directions is the directional derivative of f at the point $(-2, 0)$ equal to zero?

6. This question is about critical points of the function

$$f(x, y) = x^2 + kxy + y^2$$

   for varying values of the constant k.

   (a) Show that for any value of k, the point $(0, 0)$ is a critical point of the function f.
   (b) Find the Hessian matrix of f at the critical point $(0, 0)$. (Some of the entries of the matrix will depend on k.)
   (c) In each of the following cases, pick a value for k such that the critical point at $(0, 0)$ is of the given type:
      i. local minimum;
      ii. saddle point.
   (d) What type of critical point is $(0, 0)$ in the case $k = 2$? (Hint: the second-derivative test does not work in this case. You need to find another way to determine what type of critical point it is.)

7. The discrete random variable X has the following distribution:

$$P(X = 1) = 2/5,\ P(X = 2) = 2/5,\ P(X = 3) = 0,\ P(X = 4) = 1/5.$$

   (a) Find the expectation and variance of X.

   (b) We measure X three times independently. Find the probability that $X = 4$ for exactly two of those measurements.

   (c) Use the Central Limit Theorem to find the approximate probability that the average of 120 independent measurements of X is larger than 2.2.

8. The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{for } 1 \le x \le 2; \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the cumulative density function $F(x) = P(X \le x)$.

   (b) Find the value x such that $P(X \le x) = P(X \ge x) = 0.5$.

   (c) Calculate the expectation of X.

   (d) The continuous random variable Y has probability density function given by

$$f(y) = \begin{cases} 1/4 & \text{for } 0 \le x \le 4; \\ 0 & \text{otherwise.} \end{cases}$$

   Which of X and Y would you expect to have the larger variance? Explain your answer.

9. A three-sided coin has sides marked $0, 1, 2$ and lands on each side with probability $1/3$. If I flip two such coins, let X be the random variable that measures my total score, and let Y be the random variable given by $Y = 1$ if both coins show the same side and $Y = 2$ if they show different sides.

   (a) Find $P(Y = 1)$.

   (b) Find $P(X = 3)$.

   (c) Find EX and EY. (Make sure you show all your working.)

   (d) Show that X and Y are *not* independent. (It is not enough to explain this only in words. You must do a calculation that shows they are not independent.)

10. An experiment was carried out to estimate the probability that a certain coin will show heads. The coin was flipped 40 times and came up heads 13 times.

   (a) Find a 95% confidence interval for the probability p that this coin shows heads. (Even if you aren't able to simplify the answer, you will get partial credit for showing the method.)

   (b) In fact, it is discovered that the probability of getting a head is 0.4. Using the normal approximation to the binomial distribution (and the histogram correction), explain how to get an estimate for the probability that you get 13 heads when flipping the coin 40 times. (You can leave your answer in terms of expressions of the form $P(Z \le z)$ where Z has a standard normal distribution.)