Math 107: Calculus II, Spring 2008: Midterm Exam III Solutions

1. *Suppose we flip three fair coins. Let A stand for the event that the first two coins are both heads. Let B stand for the event that the third coin is different from the second coin.*

   (a) *Find $P(A)$ and $P(B)$.*

   (b) *Find the conditional probability $P(A|B)$.*

   (c) *Are the events A and B independent? (Explain your answer.)*

   (a) $P(A) = P(HHT) + P(HHH) = 2/8 = 1/4$, $P(B) = P(HHT) + P(HTH) + P(THT) + P(TTH) = 4/8 = 1/2$

   (b) The formula for the conditional probability is

   $$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

   $P(A \text{ and } B) = P(HHT) = 1/8$ and so we get $P(A|B) = \frac{1/8}{4/8} = 1/4$

   (c) We have $P(A \text{ and } B) = 1/8$ and $P(A)P(B) = \frac{1}{4}\frac{1}{2} = 1/8$. Since these are equal, the events A and B **are** independent.

2. *A game involves four cards with numbers on as pictured below:*

   $\boxed{1}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$

   The game consists of picking one card at random, noting the number on it, then picking a second card at random from the remaining three and noting its number.

   (a) *List the outcomes for this game and calculate their probabilities.*

   (b) *Let X be the random variable that equals the sum of the numbers on the two cards picked. Find the expectation and variance of X.*

   (a) $P(1, 2) = 1/4$, $P(2, 1) = 1/4$, $P(2, 2) = 1/2$

   (b) The distribution of X is:

   $$P(X = 3) = P(2, 1) + P(1, 2) = 1/2; P(X = 4) = P(2, 2) = 1/2.$$

   Therefore

   $$EX = 3 \times \frac{1}{2} + 4 \times \frac{1}{2} = \frac{7}{2}$$

   and hence

   $$\text{Var}(X) = (3 - 7/2)^2 \times \frac{1}{2} + (4 - 7/2)^2 \times \frac{1}{2} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

3. *The random variable X is binomially distributed with 3 repetitions and probability of success 1/3. The random variable Y is binomially distributed with 300 repetitions and probability of success 1/4.*

   (a) *Find $P(X = 1)$. (Give your answer as a fraction.)*

   (b) *Find $E(X + Y)$.*

   (c) *You are told that $E(XY) = 100$. Are X and Y independent? Explain your answer.*

   (d) *Use the normal approximation to the binomial distribution (including the histogram correction) to find an estimate for $P(Y \leq 67)$.*

   (a) $P(X = 1) = \binom{3}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 = 3\frac{1}{3}\frac{4}{9} = \frac{4}{9}.$

   Note that the binomial coefficient $\binom{3}{1}$ is equal to $\frac{3!}{1!2!} = \frac{6}{1 \times 2} = 3.$

   (b) $E(X + Y) = EX + EY$. $EX = 3 \times \frac{1}{3} = 1$, $EY = 300 \times \frac{1}{4} = 75$. Therefore, $E(X + Y) = 1 + 75 = 76$.

   (c) If $E(XY) = 100$, then $E(XY) \neq (EX)(EY)$ which is $1 \times 75 = 75$. Therefore, X and Y are **not** independent.

   (d) The normal approximation to the binomial distribution says that we can pretend Y has a normal distribution. The mean and standard deviation of that normal distribution are the same as the mean and standard deviation of Y as a binomial distribution: $\mu = E(Y) = np = 75$ and $\sigma = \sqrt{\mathrm{Var}(Y)} = \sqrt{np(1-p)} = \sqrt{300 \times \frac{1}{4} \times \frac{3}{4}} = \sqrt{900/16} = 30/4 = 7.5$.

   Note: the formula $\sigma = \sqrt{\mathrm{Var}(X)/n}$ does **not** apply in this question. That formula is only when you are using the Central Limit Theorem to look at the average of n independent measurements of a random variable. That is a separate sort of question (as in question 5).

   Since we want $P(Y \leq 67)$, the histogram correction says we should find $P(Y \leq 67.5)$. (This is not 66.5 because we want to include everything that rounds to 67. So the region from 66.5 to 67.5 should be included in the calculation.)

   Therefore we have

$$
\begin{aligned}
P(Y \leq 67.5) &\simeq P(Z \leq \frac{67.5 - 75}{7.5}) \\
&= P(Z \leq -1) = 1 - P(Z \leq 1) = 1 - 0.8413 \\
&= 0.1587.
\end{aligned}
$$

4. *The continuous random variable X has probability density function (pdf) given by*

$$
f(x) = \begin{cases} 1 - \frac{1}{2}x & \textit{if } 0 \leq x \leq 2; \\ 0 & \textit{otherwise.} \end{cases}
$$

(a) *Find* $P(1 \leq X \leq 2)$.

(b) *Find the expectation of* X.

(c) *Find the cumulative distribution function (cdf) of* X.

(a) $P(1 \leq X \leq 2) = \int_1^2 (1 - \frac{1}{2}x)\ dx = \left[x - \frac{1}{4}x^2\right]_1^2 = (2-1) - (1 - 1/4) = 1/4.$

(b) $EX = \int_0^2 x(1 - \frac{1}{2}x)\ dx = \int_0^2 (x - \frac{1}{2}x^2)\ dx = \left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_0^2 = 2 - 8/6 = 2/3.$

(c) The cdf is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ \int_0^x f(x)\ dx & \text{if } 0 \leq x \leq 2; \\ 1 & \text{if } x > 2. \end{cases}$$

Working out the integral here, we get

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ x - \frac{1}{4}x^2 & \text{if } 0 \leq x \leq 2; \\ 1 & \text{if } x > 2. \end{cases}$$

5. *The random variable* X *has a normal distribution with mean* $\mu = 8.4$ *and standard deviation* $\sigma = 4$.

(a) *Sketch a graph of the pdf of* X. *(You should label the* x*-axis as accurately as possible, but you do not have to label the* y*-axis.)*

(b) *Find* $P(X \leq 8)$.

(c) *Suppose* $\overline{X}$ *is the average of 25 independent measurements of* X. *Use the Central Limit Theorem to find an estimate for* $P(\overline{X} \leq 8)$.

(a) The pdf should be a bell-shaped curve with peak at $x = 8.4$ and with the 'hill' roughly spreading 4 units either side of the peak.

(b) $P(X \leq 8) = P(Z \leq (8-8.4)/4) = P(Z \leq -0.1) = 1 - P(Z \leq 0.1) = 1 - 0.5398 = 0.4602$.

(c) The Central Limit Theorem tells us that $\overline{X}$ is roughly a normal distribution with mean $\mu = EX = 8.4$ and standard deviation $\sigma = \sqrt{\text{Var}(X)/n}$. The variance of X is the square of the standard deviation of X which is $4^2 = 16$. Therefore $\sigma = \sqrt{16/25} = 4/5 = 0.8$. Hence we have

$$\begin{aligned} P(\overline{X} \leq 8) &\simeq P(Z \leq (8 - 8.4)/0.8) \\ &= P(Z \leq -0.5) = 1 - P(Z \leq 0.5) = 1 - 0.6915 \\ &= 0.3085. \end{aligned}$$