Math 107: Calculus II, Spring 2008: Midterm I
Practice Exam II Solutions

Solutions

1. (a) *Find the general solution to the following separable differential equation*

$$\frac{dx}{dt} = tx - t.$$

(b) *Find the specific solution to the differential equation from part (a) that has*

$$x(0) = -1.$$

(Write your answers in the form $x(t) = \dots$*.)*

(a) We first separate the variables to get:

$$\int \frac{1}{x-1}\, dx = \int t\, dt.$$

Integrating both sides gives

$$\ln|x-1| = t^2/2 + c.$$

We now solve for x:

$$|x-1| = e^{t^2/2+c}$$
$$x - 1 = \pm e^{t^2/2+c}$$
$$x = 1 \pm e^{t^2/2+c}$$

We must also remember the 'missing solution' which is $x = 1$, so the full general solution is

$$\boxed{x(t) = 1 \pm e^{t^2/2+c} \quad \text{or} \quad x(t) = 1}$$

where c is an arbitrary constant. To simplify this expression, we can rewrite it is

$$\boxed{x(t) = 1 + Ae^{t^2/2}}$$

where A is a new constant.

(b) We substitute $x = -1$, $t = 0$ into the formula. This gives

$$-1 = 1 + A$$

and so $A = -2$. Therefore the specific solution is

$$\boxed{x(t) = 1 - 2e^{t^2/2}.}$$

A common mistake was to find the value of the other constant to be $c \ln 2$ which is correct, but to leave the $\pm$ in the answer. Since you are being asked for a specific solution, you have to say whether it is plus or minus. It cannot be both. In this case, the correct sign is minus.

2. *The giant panda population at time t (measured in years) is given by the function $P(t)$ (measured in thousands – i.e. $P = 1$ means there are 1000 pandas). This function satisfies the following autonomous differential equation*

$$\frac{dP}{dt} = (P-2)(10-P).$$

   (a) *What are the equilibrium solutions to this differential equation?*

   (b) *Use the first derivative test to find the stability of the equilibrium solutions.*

   (c) *Sketch the solutions to this differential equation.*

   (d) *What is the smallest population of pandas that can survive, that is, not die away to zero?*

   (a) The equilibrium solutions are the values of P that make $dP/dt = 0$, i.e.

$$\boxed{P = 2,\ P = 10.}$$

   (b) The first derivative test for stability says that we should look at $g'(P)$ at the equilibria, where $g(P)$ is the function $(P-2)(10-P)$. Multiplying this out we get

$$g(P) = -P^2 + 12P - 20$$

   and so

$$g'(P) = -2P + 12.$$

   We now evaluate at each of the equilibria:

$$g'(2) = 8 > 0$$

   so $P = 2$ is an **unstable** equilibrium and

$$g'(10) = -8 < 0$$

   so $P = 10$ is a **stable** equilibrium.

   (c) (See sheet of graphs at the end.)

   (d) If we have an initial condition with $P < 2$ then the population will die away to zero. If $P = 2$ the population will stay constant. Therefore, the smallest population that can survive is **2000 pandas**.

3. *This question is about the following system of equations*

$$x + cy = 4$$
$$cx + 4y = 8$$

where c is an unknown number.

(a) *Find the inverse of the matrix* $\begin{pmatrix} 1 & c \\ c & 4 \end{pmatrix}$. *Make sure you say for what values of* c *this inverse exists.*

(b) *For what values of* c *does the above system of equations have exactly one solution? Explain how you would use your answer to part (a) to solve the equations in this case. (You do not need to actually find the solution.)*

(c) *Find* all *the solutions to this system of equations in each of the following cases:*

i. $c = 2$;

ii. $c = -2$.

(a) Using the formula for the inverse matrix we get

$$\boxed{\frac{1}{4-c^2}\begin{pmatrix} 4 & -c \\ -c & 1 \end{pmatrix}}$$

This exists whenever the determinant is nonzero, i.e. when $\mathbf{c} \neq \pm \mathbf{2}$.

(b) The system of equations has exactly one solution precisely when the above inverse matrix has an inverse, i.e. for $\mathbf{c} \neq \pm \mathbf{2}$. In this case we can solve the equations by using the formula $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, i.e.

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4-c^2}\begin{pmatrix} 4 & -c \\ -c & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \end{pmatrix}.}$$

(c) i. The equations are now $x + 2y = 4$ and $2x + 4y = 8$. These are equivalent so there are infinitely many solutions. **Any pair** x, y **satisfying** $x + 2y = 4$ **is a solution.**

ii. The equations are $x - 2y = 4$ and $-2x + 4y = 8$ which are inconsistent (multiplying the first by -2 gives $-2x + 4y = -8$). Therefore **there are no solutions**.

4. (a) *The matrix*

$$\begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}$$

has only one eigenvalue k. *What is it?*

(b) *Find an eigenvector* $\begin{pmatrix} u \\ v \end{pmatrix}$ *corresponding to the eigenvalue you found in part (a).*

(c) *By substituting in (using the values of* k, u, v *that you found in parts (a) and (b)), show that*

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{kt}\begin{pmatrix} u \\ v \end{pmatrix}$$

is a solution to the following system of differential equations

$$\frac{dx}{dt} = -2x + y; \quad \frac{dy}{dt} = -x - 4y.$$

(a) To find the eigenvalue, we take the determinant of $\mathbf{A} - k\mathbf{I}$:

$$\det \begin{pmatrix} -2-k & 1 \\ -1 & -4-k \end{pmatrix} = (-2-k)(-4-k)+1 = k^2+6k+9 = (k+3)^2.$$

Therefore the eigenvalue is $\mathbf{k = -3}$.

(b) To find an eigenvector, we have to solve the equation

$$\begin{pmatrix} -2-k & 1 \\ -1 & -4-k \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which with $k = -3$ is

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

These equations work out as $u + v = 0$ and $-u - v = 0$. Note that these are equivalent which should always be true when finding an eigenvector by this method. One solution is given by $u = 1$ and $v = -1$, so an eigenvector is

$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.}$$

(c) To show that something is a solution to a system of differential equations, you have to substitute it into the equations and show that you get equality. Some people showed that their eigenvector was really an eigenvector, which was not what the question asked for. The solution in this case is

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

or $x = e^{-3t}$, $y = -e^{-3t}$. If we substitute these into the differential equations we get

$$\frac{dx}{dt} = -3e^{-3t}; \; -2x + y = -2e^{-3t} + (-e^{-3t}) = -3e^{-3t}$$

which are equal, and

$$\frac{dy}{dt} = 3e^{-3t}; \; -x - 4y = -(e^{-3t}) - 4(-e^{-3t}) = 3e^{-3t}$$

which are equal. So this is indeed a solution.

5. *The matrix* $\mathbf{A}$ *satisfies the following equations:*

$$\mathbf{A} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$
$$\mathbf{A} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

(a) *Use this information to write down the two eigenvalues and corresponding eigenvectors of the matrix* $\mathbf{A}$*.*

(b) *Find the solution to the system of differential equations*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}$$

that satisfies

$$x(0) = 2; \quad y(0) = 3.$$

(Write your answer in the form $x(t) = \ldots, y(t) = \ldots$*.)*

(c) *Is the equilibrium of this system of equations stable or unstable?*

(a) A lot of people wasted a lot of time on this question. **You do not need to find the matrix A!** The equations given in the question tell you immediately what the eigenvectors and eigenvalues of the matrix are, without needing to find the matrix. The first equation tells you that

$$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which means that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue 2. Similarly, the second equation tells you that $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue -2.

(b) Using the answer to part (a), the general solution to this system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

To find the specific solution, we substitute in the initial conditions $x(0) = 2$ and $y(0) = 3$ to get

$$2 = c_1 + c_2$$
$$3 = 0 - 3c_2$$

This means that $c_2 = -1$ and so $c_1 = 3$. Putting this all together we get

$$\boxed{x(t) = 3e^{2t} - e^{-2t}; \quad y(t) = 3e^{-2t}.}$$

(c) Since one of the eigenvalues is positive, this is an **unstable** equilibrium.