Math 107: Calculus II, Spring 2008: Midterm I: Practice exam 2
(This is the actual exam from Fall 2006)

The following instructions will appear on the exam.

1. There are five questions. Each is worth 20 points.
2. **Do not open your booklet until told to begin.** The exam will be 50 minutes long.
3. You may **not** use calculators, books, notes or any other paper. Write all your answers on this booklet. Additional paper is available if required.
4. **You must show all your working and explain your answers clearly to obtain full credit!**
5. **Read the questions carefully!** Some questions only require an answer, others require particular explanations. If in doubt, write more!

Questions

1. (a) Find the general solution to the following separable differential equation

$$\frac{dx}{dt} = tx - t.$$

   (b) Find the specific solution to the differential equation from part (a) that has

$$x(0) = -1.$$

   (Write your answers in the form $x(t) = \ldots$.)

2. The giant panda population at time t (measured in years) is given by the function $P(t)$ (measured in thousands – i.e. $P = 1$ means there are 1000 pandas). This function satisfies the following autonomous differential equation

$$\frac{dP}{dt} = (P-2)(10-P).$$

   (a) What are the equilibrium solutions to this differential equation?
   (b) Use the first derivative test to find the stability of the equilibrium solutions.
   (c) Sketch the solutions to this differential equation.
   (d) What is the smallest population of pandas that can survive, that is, not die away to zero?

3. This question is about the following system of equations

$$x + cy = 4$$
$$cx + 4y = 8$$

where c is an unknown number.

(a) Find the inverse of the matrix $\begin{pmatrix} 1 & c \\ c & 4 \end{pmatrix}$. Make sure you say for what values of c this inverse exists.

(b) For what values of c does the above system of equations have exactly one solution? Explain how you would use your answer to part (a) to solve the equations in this case. (You do not need to actually find the solution.)

(c) Find *all* the solutions to this system of equations in each of the following cases:

i. $c = 2$;

ii. $c = -2$.

4. (a) The matrix

$$\begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}$$

has only one eigenvalue k. What is it?

(b) Find an eigenvector $\begin{pmatrix} u \\ v \end{pmatrix}$ corresponding to the eigenvalue you found in part (a).

(c) By substituting in (using the values of k, u, v that you found in parts (a) and (b)), show that

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{kt} \begin{pmatrix} u \\ v \end{pmatrix}$$

is a solution to the following system of differential equations

$$\frac{dx}{dt} = -2x + y; \qquad \frac{dy}{dt} = -x - 4y.$$

5. The matrix $\mathbf{A}$ satisfies the following equations:

$$\mathbf{A} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$
$$\mathbf{A} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

(a) Use this information to write down the two eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A}$.

(b) Find the solution to the system of differential equations

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

that satisfies

$$x(0) = 2; \quad y(0) = 3.$$

(Write your answer in the form $x(t) = \ldots, y(t) = \ldots$.)

(c) Is the equilibrium of this system of equations stable or unstable?