TEST 1 (03/08/2013, MATH 107, CALCULUS II (BIO))

Name: **Section:** **Score:**

In agreeing to take this exam, you are implicitly agreeing to act with fairness and honesty.

Problems/Points	1/10	2/20	3/10	4/20	5/20	6/20
Scores						

1. (10 points) Consider a 2×2 matrix

$$A = \begin{bmatrix} 4 & 5 \\ -1 & -2 \end{bmatrix}$$

(a) Compute $\mathrm{tr}(A)$ and $\det(A)$.
(b) Find the eigenvalues of A.

Proof. (a) By definition we have

$$\begin{aligned} \mathrm{tr}(A) &= 4 + (-2) = 2, \\ \det(A) &= 4 \times (-2) - 5 \times (-1) = -8 + 5 = -3. \end{aligned}$$

(b) Suppose that λ is an eigenvalue of A. Then

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0.$$

From part (a), we obtain

$$0 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1);$$

thus $\lambda = -1$ or $\lambda = 3$. □

2. (20 points) Solve the system of linear equations

$$\begin{aligned} y + x &= 3 \\ z - y &= -1 \\ x + 2z &= 4 \end{aligned}$$

Proof.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & -1 \\ 1 & 0 & 2 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Hence $z = 2$, $y = 3$, and $x = 0$. □

3. (10 points) Compute the improper integral

$$\int_1^{e^4} \frac{dx}{x\sqrt{\ln x}}.$$

Proof. Let $u = \ln x$. Then $du = \frac{dx}{x}$ and

$$\begin{aligned}\int_0^{e^4} \frac{dx}{x\sqrt{\ln x}} &= \int_0^4 \frac{du}{\sqrt{u}} = \int_0^4 u^{-1/2}du \\ &= 2u^{1/2}\Big|_0^4 = 2(\sqrt{4} - 0) = 4.\end{aligned}$$

□

4. (20 points) Consider a 2×2 matrix

$$A = \begin{bmatrix} 3 & -2 \\ 7 & -5 \end{bmatrix}$$

(a) Find the inverse matrix A^{-1} and its determinant $\det(A^{-1})$.

(b) Define a to be the number $\det(A^{-1})$. Suppose that the volume $V(t)$ of a cell at time t changes according to

$$\frac{dV}{dt} = \sin(at) \quad \text{with } V(0) = 3.$$

Find $V(t)$.

Proof. (a) $\det(A) = 3 \times (-5) - 7 \times (-2) = -15 + 14 = -1$. So

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 7 & -3 \end{bmatrix}$$

and

$$\det(A^{-1}) = 5 \times (-3) - 7 \times (-2) = -15 + 14 = -1.$$

(b) According to part (a), $a = \det(A^{-1}) = -1$ and

$$V(t) = V(0) + \int_0^t -\sin u\, du = 3 + \int_0^t -\sin u\, du = 3 + \cos u \Big|_0^t = 2 + \cos t.$$

□

5. (20 points) Consider three vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

(a) Compute the dot products $a := \mathbf{x} \cdot \mathbf{y}$ and $b := \mathbf{x} \cdot \mathbf{z}$.

(b) Let $P = (1, -1)$ be the point in $\mathbf{R}^2$ corresponding to the vector $\mathbf{x}$. Find the line that passes through this point P and is perpendicular to the vector

$$\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$$

where a, b are defined in (a).

Proof. (a)

$$a = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1 \times 2 + (-1) \times 3 = 2 - 3 = -1,$$

$$b = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \end{bmatrix} = 1 \times (-1) + (-1) \times 4 = -1 - 4 = -5.$$

(b) By part (a),

$$\mathbf{n} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

and hence the line is given by

$$0 = \begin{bmatrix} x - 1 \\ y - (-1) \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -5 \end{bmatrix} = (-1) \times (x-1) + (-5) \times (y+1) = -x - 5y - 4,$$

namely, $x + 5y + 4 = 0$. □

6. (20 points) (a) Solve the differential equation

$$\frac{dy}{dx} = (y-2)(y+1), \quad y(0) = 3.$$

(b) Find the equilibria of the above differential equation and discuss the stability of each equilibrium.

Proof. (a) Write

$$\frac{dy}{(y-2)(y+1)} = dx \Longrightarrow \int \frac{dy}{(y-2)(y+1)} = \int dx.$$

Let

$$\frac{1}{(y-2)(y+1)} = \frac{A}{y-2} + \frac{B}{y+1}$$

then

$$\frac{1}{(y-2)(y+1)} = \frac{(A+B)y + (A-2B)}{(y-2)(y+1)}.$$

Consequently,

$$A + B = 0, \quad A - 2B = 1 \Longrightarrow A = \frac{1}{3}, \ B = -\frac{1}{3}.$$

Substituting $A = 1/3$ and $B = -1/3$ into above implies

$$\int \frac{1}{3}\left(\frac{1}{y-2} - \frac{1}{y+1}\right) dy = x + C \Longrightarrow \frac{1}{3}\ln\left|\frac{y-2}{y+1}\right| = x + C,$$

or

$$\frac{y-2}{y+1} = C'e^{3x} \Longrightarrow y = \frac{2 + C'e^{3x}}{1 - C'e^{3x}}.$$

Since $y(0) = 3$, it follows that

$$3 = \frac{2 + C'}{1 - C'} \Longrightarrow C' = \frac{1}{4}$$

and therefore

$$y = \frac{8 + e^{3x}}{4 - e^{3x}}.$$

(b) Let $g(y) = (y-2)(y+1) = y^2 - y - 2$, then $g'(y) = 2y - 1$. The equilibria are $\hat{y}_1 = -1$ or $\hat{y}_2 = 2$.

If $\hat{y}_1 = 1$, then $g'(\hat{y}_1) = 2 \times (-1) - 1 = -3 < 0$; if $\hat{y}_2 = 2$, then $g'(\hat{y}_2) = 2 \times 2 - 1 = 3 > 0$. Hence -1 is locally stable and 2 is unstable. $\square$