PRACTICE EXAM II

YI LI

1. Consider the function

$$f(x,y) = \sqrt{4 - x^2 - y^2}.$$

(a) Find the largest possible domain and the corresponding range of $f(x,y)$.
(b) Find the level curve of f.
(c) Compute $f_x(x,y)$ and $f_y(x,y)$

Proof. (a) By definition, we must have $4 - x^2 - y^2 \geq 0$ or equivalently $x^2 + y^2 \leq 4$, and then the largest possible domain of f is

$$D := \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \leq 4\}$$

the closed disk of radius 2 centered at $(0,0)$. The range of f now is $[0,2]$.

(b) The level curve of f is determined by $f(x,y) = c$, where c is a nonnegative constant (because of the range of f). Thus the level curve of f is $x^2 + y^2 = 4 - c^2$.

(c) Compute

$$f_x(x,y) = \frac{-2x}{2\sqrt{4 - x^2 - y^2}} = \frac{-x}{\sqrt{4 - x^2 - y^2}}, \quad f_y(x,y) = \frac{-y}{\sqrt{4 - x^2 - y^2}}.$$

□

2. (a) Compute

$$\lim_{(x,y)\to(-1,-2)} \frac{x^2 - y^2}{2xy + 2}.$$

(b) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy + xy^2}{x^2 + y^2}$$

does not exist.

(c) Show that

$$f(x,y) = \begin{cases} \frac{xy + xy^2}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0) \end{cases}$$

is discontinuous at $(0,0)$.

Proof. (a) By basic laws

$$\lim_{(x,y)\to(-1,-2)} \frac{x^2 - y^2}{2xy + 2} = \frac{(-1)^2 - (-2)^2}{2(-1)(-2) + 2} = \frac{1 - 4}{4 + 2} = \frac{-3}{6} = \frac{-1}{2}.$$

(b) For the lines $C_m : y = mx$, where $m \neq 0$, we have

$$\lim_{(x,y)\to(0,0) \text{ along } C_m} \frac{xy + xy^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x(mx) + x(mx)^2}{x^2 + (mx)^2} = \lim_{x\to 0} \frac{mx^2 + m^2x^3}{x^2 + m^2x^2} = \frac{m}{1+m^2}.$$

For different choice of m, we get different limit; thus the above limit does not exist.

(c) Since $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist by part (b), it follows that $f(x,y)$ is discontinuous at $(0,0)$. □

3. Let

$$f(x,y) = x^3 \cos y.$$

Compute $f_x(x,y), f_y(x,y), f_{xx}(x,y), f_{xy}(x,y), f_{yx}(x,y), f_{yy}(x,y)$.

Proof. Compute

$$f_x(x,y) = 3x^2 \cos y, \quad f_y(x,y) = -x^3 \sin y.$$

Moreover

$$\begin{aligned} f_{xx}(x,y) &= 6x \cos y, \\ f_{xy}(x,y) &= f_{yx}(x,y) = -3x^2 \sin y, \\ f_{yy}(x,y) &= -x^3 \cos y. \end{aligned}$$

□

4. Let

$$\mathbf{h}(x,y) = \begin{bmatrix} e^{4x-\sqrt{6}y} \\ e^{\sqrt{6}x-y} \end{bmatrix}$$

(a) Find the Jacobi matrix $(D\mathbf{h})(x,y)$.
(b) Compute $(D\mathbf{h})(0,0)$.
(c) Let us denote by A the 2×2 matrix $(D\mathbf{h})(0,0)$. Find the eigenvalues and eigenvectors of A.

Proof. Let $f(x,y) = e^{4x-\sqrt{2}y}$ and $g(x,y) = e^{\sqrt{2}x-y}$.

(a) Then

$$(D\mathbf{h})(x,y) = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix} = \begin{bmatrix} 4e^{4x-\sqrt{6}y} & -\sqrt{6}e^{4x-\sqrt{6}y} \\ \sqrt{6}e^{\sqrt{6}x-y} & -e^{\sqrt{6}x-y} \end{bmatrix}$$

(b) By (a) we obtain

$$(D\mathbf{h})(0,0) = \begin{bmatrix} 4 & -\sqrt{6} \\ \sqrt{6} & -1 \end{bmatrix}$$

(c) $\text{tr}(A) = 4 + (-1) = 3$ and $\det(A) = 4(-1) - (-\sqrt{6})\sqrt{6}) = -4 + 6 = 2$. An eigenvalue λ satisfies

$$0 = \lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2).$$

Hence $\lambda_1 = 1$ and $\lambda_2 = 2$.

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 1$, then

$$0 = (A - \lambda_1 I_2)\mathbf{x} = \begin{bmatrix} 3 & -\sqrt{6} \\ \sqrt{6} & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - \sqrt{6}x_2 \\ \sqrt{6}x_1 - 2x_2 \end{bmatrix}$$

so that $3x_1 = \sqrt{6}x_2$ while the second equation is canonically identical to the first one. Hence the eigenvector associated to $\lambda_1 = 1$ is

$$t \begin{bmatrix} \sqrt{6} \\ 3 \end{bmatrix}, \quad t \in \mathbf{R} \setminus \{0\}.$$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector for $\lambda_2 = 2$, then

$$0 = (A - \lambda_2 I_2)\mathbf{x} = \begin{bmatrix} 2 & -\sqrt{6} \\ \sqrt{6} & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - \sqrt{6}x_2 \\ \sqrt{6}x_1 - 3x_2 \end{bmatrix}$$

so that $2x_1 = \sqrt{6}x_2$ while the second equation is canonically identical to the first one. Hence the eigenvector associated to $\lambda_2 = 2$ is

$$t \begin{bmatrix} \sqrt{6} \\ 2 \end{bmatrix}, \quad t \in \mathbf{R} \setminus \{0\}.$$

□

5. Find a linear approximation to

$$\mathbf{f}(x, y) = \begin{bmatrix} (x-y)^2 \\ 2x^2 y \end{bmatrix}$$

at $(2, -3)$.

Proof. The linear approximation is given by

$$\begin{aligned} L(x,y) &= \mathbf{f}(2,-3) + Df(2,-3) \begin{bmatrix} x-2 \\ y-(-3) \end{bmatrix} \\ &= \begin{bmatrix} 25 \\ -24 \end{bmatrix} + \begin{bmatrix} 2(x-y) & 2(x-y)(-1) \\ 4xy & 2x^2 \end{bmatrix}_{(x,y)=(2,-3)} \begin{bmatrix} x-2 \\ y+3 \end{bmatrix} \\ &= \begin{bmatrix} 25 \\ -24 \end{bmatrix} + \begin{bmatrix} 10 & -10 \\ -24 & 8 \end{bmatrix} \begin{bmatrix} x-2 \\ y+3 \end{bmatrix} \\ &= \begin{bmatrix} 10x - 10y - 25 \\ -24x + 8y + 48 \end{bmatrix}. \end{aligned}$$

□

6. (a) Find the gradient of

$$f(x,y) = \ln\left(\frac{x}{y} + \frac{y}{x}\right).$$

(b) Compute the directional derivative of

$$f(x,y) = 2xy^3 - 3x^2 y$$

at $(1,-1)$ in the direction $\begin{bmatrix}3\\1\end{bmatrix}$.

(c) Compute the directional derivative of

$$f(x,y) = 2x^2y - 3x$$

at the point $P = (2,1)$ in the direction of the point $Q = (3,2)$.

(d) Find a unit vector that is normal to the level curve of the function

$$f(x,y) = x^2 - y^3$$

at the point $(1,3)$.

Proof. (a) Since

$$f_x(x,y) = \frac{1}{\frac{x}{y}+\frac{y}{x}}\left(\frac{1}{y} - \frac{y}{x^2}\right) = \frac{x^2-y^2}{x(x^2+y^2)}$$

and $f(x,y)$ is symmetric in x and y, we arrive at

$$\nabla f(x,y) = \begin{bmatrix}\frac{x^2-y^2}{x(x^2+y^2)}\\ \frac{y^2-x^2}{y(x^2+y^2)}\end{bmatrix}$$

(b) The gradient of f is

$$\nabla f(x,y) = \begin{bmatrix}2y^3 - 6xy\\ 6xy^2 - 3x^2\end{bmatrix}$$

and the gradient vector of f at $(1,-1)$ is then equal to

$$\nabla f(1,-1) = \begin{bmatrix}-2+6\\ 6-3\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}.$$

Since the vector $\begin{bmatrix}3\\1\end{bmatrix}$ is not unit, we normalize it by

$$\mathbf{u} = \frac{1}{\left|\begin{bmatrix}3\\1\end{bmatrix}\right|}\begin{bmatrix}3\\1\end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}.$$

Hence

$$D_{\mathbf{u}}f(1,-1) = \nabla f(1,-1)\cdot\mathbf{u} = \begin{bmatrix}4\\3\end{bmatrix}\cdot\frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix} = \frac{4\times 3 + 3\times 1}{\sqrt{10}} = \frac{15}{\sqrt{10}}.$$

(c) The vector $\overrightarrow{PQ}$ is given by

$$\begin{bmatrix}3-2\\2-1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

which is not a unit vector. Then we normalize it by

$$\mathbf{u} = \frac{1}{\left|\begin{bmatrix}1\\1\end{bmatrix}\right|}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

Compute

$$\nabla f(x,y) = \begin{bmatrix}4xy-3\\ 2x^2\end{bmatrix} \Longrightarrow \nabla f(2,1) = \begin{bmatrix}5\\8\end{bmatrix}.$$

Hence

$$D_{\mathbf{u}}f(2,1) = \nabla f(2,1) \cdot \mathbf{u} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{13}{\sqrt{2}}.$$

(d) The gradient vector of f is

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ -3y^2 \end{bmatrix}.$$

The vector that is normal to the level curve at the point $(1,3)$ is the gradient vector $\nabla f(1,3)$, that is,

$$\nabla f(1,3) = \begin{bmatrix} 2 \\ -27 \end{bmatrix}.$$

This is a nonunit vector, so that its normalization vector

$$\mathbf{u} = \frac{1}{\left\| \begin{bmatrix} 2 \\ -27 \end{bmatrix} \right\|} \begin{bmatrix} 2 \\ -27 \end{bmatrix} = \frac{1}{\sqrt{733}} \begin{bmatrix} 2 \\ -27 \end{bmatrix}$$

is the required one. □

7. Let

$$f(x,y) = -2x^2 + y^2 - 6y.$$

find all candidates for local extrema and determine the type (local maximum, local minimum, or saddle point).

Proof. Since f is differentiable on $\mathbf{R}^2$, the only critical points satisfy

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \nabla f(x,y) = \begin{bmatrix} -4x \\ 2y - 6 \end{bmatrix};$$

hence f has only one critical point $(0,3)$. Compute the Hessian matrix of f:

$$\mathbf{Hess}(f)(x,y) = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$$

Since $\det(\mathbf{Hess}(f)(0,3)) = -8 < 0$, it follows that the critical point $(0,3)$ is a saddle point. □

Department of Mathematics, Johns Hopkins University, 3400 N Charles Street, Baltimore, MD 21218, USA

E-mail address: yli@math.jhu.edu