PRACTICE EXAM I

YI LI

1. Solve the differential equation
$$\frac{dy}{dx} = e^{-3x}y^2.$$

Proof. dividing by y^2 on both sides implies
$$\frac{dy}{y^2} = e^{-3x}dx;$$
hence
$$\int \frac{dy}{y^2} = \int e^{-3x}dx \Longrightarrow \frac{-1}{y} = \int e^{-3x}dx.$$
Let $u = -3x$ and then $du = -3dx$. So
$$\frac{1}{y} = \frac{1}{3}\int e^u du \Longrightarrow y = \frac{3}{e^u + C}.$$
□

2. (a) Use partial fractions to show that
$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$
where $a > 0$.

(b) Solve the differential equation
$$\frac{dy}{dx} = y^2 - 4$$
that passes through $(0, 4)$.

Proof. Write
$$\frac{1}{u^2 - a^2} = \frac{A}{u-a} + \frac{B}{u+a}.$$
Then
$$\frac{1}{u^2 - a^2} = \frac{A(u+a) + B(u-a)}{u^2 - a^2} \Longrightarrow 1 = (A+B)u + (A-B)a;$$
since the above identity holds for any u, we must obtain
$$A + B = 0, \quad (A - B)a = 1 \Longrightarrow A = \frac{1}{2a} \quad B = -\frac{1}{2a}.$$
Consequently,
$$\frac{1}{u^2 - a^2} = \frac{1}{2a}\frac{1}{u-a} - \frac{1}{2a}\frac{1}{u+a} \Longrightarrow \int \frac{du}{u^2 - a^2} = \frac{1}{2a}\int \left(\frac{1}{u-a} - \frac{1}{u+a}\right) du.$$

(a) By the above formula we arrive at

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\left(\ln|u-a| - \ln|u+a|\right) + C = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C.$$

(b) Applying (a) to the differential equation, we find that

$$\int dx = \int \frac{dy}{y^2 - 4} = \int \frac{dy}{y^2 - 2^2} \Longrightarrow x = \frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| + C.$$

Since $y(0) = 2$, it follows that

$$0 = \frac{1}{4}\ln\left|\frac{4-2}{4+2}\right| + C \Longrightarrow C = \frac{1}{4}\ln 3$$

and hence

$$x = \frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| + \frac{1}{4}\ln 3.$$

□

3. (a) Show that

$$\frac{1}{\sqrt{1+x^2}} \geq \frac{1}{2x}$$

for all $x \geq 1$.

(b) Use the result in (a) to show that

$$\int_1^\infty \frac{1}{\sqrt{1+x^2}} dx$$

is divergent.

Proof. (a) Taking the square on both sides of the inequality, we suffice to prove

$$\frac{1}{1+x^2} \geq \frac{1}{4x^2}$$

or

$$4x^2 \geq 1 + x^2 \Longleftrightarrow 3x^2 \geq 1.$$

But $x \geq 1$, we always have $3x^2 \geq 3 \times 1^2 = 3 > 1$.

(b) According to (a),

$$\int_1^\infty \frac{1}{\sqrt{1+x^2}} dx \geq \int_1^\infty \frac{dx}{2x} = \frac{1}{2}\ln x\Big|_1^\infty = \infty;$$

therefore the above improper integral is divergent. □

4. (a) Find the equilibria of the differential equation

$$\frac{dy}{dx} = (y-1)y(y+1).$$

(b) Compute the eigenvalues associated with each equilibrium and discuss the stability of the equilibria.

Proof. Let

$$g(y) := (y-1)y(y+1) = y^3 - y.$$

(a) The equilibria are $y_1 = -1$, $y_2 = 0$, and $y_3 = 1$.
(b) Compute $g'(y) = 3y^2 - 1$. Then

$$\begin{aligned} g'(y_1) = 2 > 0 &\implies y_1 \text{ is unstable,} \\ g'(y_2) = -1 < 0 &\implies y_2 \text{ is locally stable,} \\ g'(y_3) = 2 > 0 &\implies y_3 \text{ is unstable.} \end{aligned}$$

□

5. Consider matrices

$$A = \begin{bmatrix} 4 & -1 \\ 8 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

(a) Find the inverse matrix A^{-1}.
(b) Compute $\operatorname{tr}(A)$ and $\det(A)$.
(c) Find the eigenvalues of A.
(d) Solve the matrix equation $AX = B$.
(e) Determine whether the matrix BC' is invertible, where C' is the transpose of C.
(f) Compute $D = A - BC'$ and determine whether the real parts of both eigenvalues are negative.
(g) Compute the dot product $B \cdot C$.

Proof. (b) $\operatorname{tr}(A) = 4 + (-1) = 3$ and $\det(A) = 4 \times (-1) - 8 \times (-1) = 4$.
(b) Since $\det(A) \neq 0$, the inverse of A exists and is given by

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} -1 & 1 \\ -8 & 4 \end{bmatrix} = \begin{bmatrix} \frac{-1}{4} & \frac{1}{4} \\ -2 & 1 \end{bmatrix}.$$

(c) If λ denotes an eigenvalue of A, then

$$0 = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = \lambda^2 - 3\lambda + 4 \implies \lambda = \frac{3 \pm i\sqrt{7}}{2}.$$

(d) According to (a), we have

$$X = A^{-1}B = \begin{bmatrix} \frac{-1}{4} & \frac{1}{4} \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{-5}{4} \\ -7 \end{bmatrix}.$$

(e) Compute

$$BC' = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -3 & -12 \end{bmatrix}.$$

Since $\det(BC') = 2 \times (-12) - 8 \times (-3) = 0$, it follows that BC' is singular to non-invertible.
(f) By (e),

$$D = \begin{bmatrix} 4 & -1 \\ 8 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 8 \\ -3 & -12 \end{bmatrix} = \begin{bmatrix} 2 & -9 \\ 11 & 11 \end{bmatrix}.$$

Since $\text{tr}(D) = 2 + 11 = 13 > 0$ and $\det(D) = 2 \times 11 - 11 \times (-9) = 121 > 0$, the real parts of both eigenvalues are not negative.

(g) $B \cot C = 2 \times 1 + (-3) \times 4 = -10$. □

6. Find the equation of the plane through $(0, 0, 0)$ and perpendicular to $[1, 0, 0]'$.

Proof. Let $P_0 = (0, 0, 0)$ and $\mathbf{n} = [1, 0, 0]'$. For any point $P = (x, y, z)$ on this plane, we must have

$$0 = \overrightarrow{P_0P} \cdot \mathbf{n} \Longrightarrow 0 = \begin{bmatrix} x - 0 \\ y - 0 \\ z - 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x.$$

Thus the plane is $\{(x, y, z) \in \mathbf{R}^3 : x = 0\}$. □

7. Solve the system of linear equation

$$\begin{aligned} 3x - 2y + z &= 4 \\ 4x + y - 2z &= -12 \\ 2x - 3y + z &= 7 \end{aligned}$$

Proof. There are lots of ways to solve this system. I chose one approach. From the first equation, we have $z = 4 - 3x + 2y$. Substituting it into other equations implies

$$\begin{aligned} 10x - 3y &= -4 \\ x + y &= -3 \end{aligned}$$

Define

$$A = \begin{bmatrix} 10 & -3 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ -3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Since $\det(A) = 13$ and

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 1 & 3 \\ -1 & 10 \end{bmatrix},$$

we get

$$X = A^{-1}B = \frac{1}{13} \begin{bmatrix} 1 & 3 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} -4 \\ -3 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -13 \\ -26 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

Thus $x = -1$ and $y = -2$; by the first equation we have $z = 3$. □

8. Find the parametric equation of the line in the x-y plane that goes through the given points $(1, -3)$ and $(4, 0)$. Then eliminate the parameter to find the equation of the line in standard form.

Proof. Define

$$P_1 = (1, -3), \quad P_2 = (4, 0).$$

Then

$$\mathbf{u} := \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The parametric equation now is given by

$$\begin{bmatrix} x-4 \\ y-0 \end{bmatrix} = t \begin{bmatrix} 3 \\ 3 \end{bmatrix} \Longrightarrow \begin{array}{l} x = 3t+4, \\ y = 3t. \end{array}$$

In other words, $y = x - 4$. □

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