Math 107 Practice Exam 2

1a.

$$y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}\ ,\ y' = x\sqrt{x^2+2}\ ,\ 1+(y')^2 = 1+x^2(x^2+2) = (x^2+1)^2.$$

$$L = \int_0^1 \sqrt{1+(y')^2}\ dx = \int_0^1 (x^2+1)\ dx = (\frac{1}{3}x^3+x)|_0^1 = \frac{4}{3}\ .$$

b. Revolve $y = \frac{1}{3}x^3$ about the x axis from $x = 0$ to $x = 1$.

$$S = 2\pi \int_0^1 \frac{1}{3}x^3\sqrt{1+x^4}\ dx = \frac{1}{6}\pi(1+x^4)^{\frac{3}{2}}|_0^1 = \frac{\pi}{6}[2^{\frac{3}{2}}-1]\ .$$

2. Let $p(x) = c\sqrt{x}$ on $[0,1]$.

a. $p(x)$ is a probability disribution if

$$\int_0^1 p(x)\ dx = c\int_0^1 \sqrt{x}\ dx = c\cdot\frac{2}{3} = 1$$

Hence $c = \frac{3}{2}$ and $p(x) = \frac{3}{2}\sqrt{x}$. Also note that the cumulative distribution function $F(x$ is given by

$$F(x) = \int_0^x p(t)\ dt = x^{\frac{3}{2}}\ .$$

b.The mean μ of the distribution is given by

$$\mu = \int_0^1 xp(x)\ dx = \frac{3}{2}\int_0^1 x^{\frac{3}{2}} = \frac{3}{2}\cdot\frac{2}{5} = \frac{3}{5}\ .$$

c.The median m of the distribution is the that value where the probability of being less than m is $\frac{1}{2}$ (i.e the cumulative distribution function satisfies $F(m) = \frac{1}{2}$) which is the same as saying that the area under the graph of $p(x)$ from 0 to m is $\frac{1}{2}$. That is,

$$F(m) = m^{\frac{3}{2}} = \frac{1}{2} \quad \text{so} \quad m = 2^{-\frac{2}{3}}\ .$$

3 .

$$\int_e^\infty \frac{dx}{x(\ln x)^2} = \lim_{L\to\infty}\int_e^L \frac{dx}{x(\ln x)^2}$$

$$= \lim_{L\to\infty} \int_e^L [-\tfrac{1}{\ln x}]_0^L$$

$$= \lim_{L\to\infty}[1-\tfrac{1}{\ln L}] = 1$$

where we have performed the integration via the substitution $u = \ln x$.

4a. The trapezoid rule says that as $n \to \infty$, $\int_a^b f(x)\,dx \approx T_n$ where

$$T_n = \frac{(b-a)}{2n}[y_0 + 2y_1 + 2y_2 + \ldots 2y_{n-1} + y_n]$$

where the partition points $x_i = a + i\frac{(b-a)}{n}$, $i = 0, 1, \ldots, n$ and $y_i = f(x_i)$.

b. $\ln 2 = \int_1^2 \frac{1}{x}\,dx$ so we take $f(x) = \frac{1}{x}$, $a = 1, b = 2, n = 5$.

$$T_5 = (.1)[1/1 + 2(1/1.2) + 2(1/1.4) + 2(1/1.6) + 2(1/1.8) + 1/2] = 0.695$$

5.We use the exponential distribution with $k = .05$; that is , $p(x) = 0.05e^{-.05x}$. As computed in my notes.

$$\mu = \int_0^\infty 0.05xe^{-.05x}\,dx = 1/.05 = 20$$

Hence the average lifespan is 20 years.

6. We use the normal distribution with $\mu = 30,000$ and standard deviation $\sigma = 2000$.

a.We need to compute the probabilty $P(25000 < X < 28000)$ (X=lifespan) by converting to the standard normal $Z = \frac{X-30,000}{2000}$. Since $\frac{25000-30000}{2000} = -\frac{5}{2} = -2.5$ and $\frac{28000-30000}{2000} = -1$ we need the area under the standard normal from $Z = -2.5$ to $Z = -1$. Since the given Z-table gives the area from 0 to Z, by the symmetry of the normal distribution, it is equivalent to find the area from $Z = 1$ to $Z = 2.5$. This is 0.4938-0.3413=0.1525. So the percentage is 15.25%.

b.To find the probability that $X > 36000$ we convert this to $Z > \frac{36000-30000}{2000} = 3$. Looking in the table we see that the probability(area) is 0.5-0.4987=0.0013 .

Z	Area from 0 to Z
0	0.0000
0.5	0.1915
1.0	0.3413
1.5	0.4332
2.0	0.4772
2.5	0.4938
3	0.4987