Math 107 Soutions to Practice Exam 1
Part I

1.

$$\begin{aligned} \int x^3 \ \ln x \ dx &= \ln x \ \frac{x^4}{4} - \int \frac{x^4}{4} \ \frac{1}{x} \\ &= \frac{x^4}{4} \ \ln x - \frac{x^4}{16} \ + C \end{aligned}$$

2.

$$\begin{aligned} \frac{1}{(x-1)(x-3)} &= \frac{A}{x-1} + \frac{B}{x-3} \\ &= \frac{-\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x-3} \end{aligned}$$

Hence,

$$\int \frac{1}{(x-1)(x-3)} \ dx = \frac{1}{2} \ln \frac{x-3}{x-1} + C$$

3. Two mg of radioactive material decays to 1.3 mg after 10 days. Write $N(t) = 2 \cdot e^{-kt}$ Then

$$e^{-10k} = \frac{1.3}{2} = .65$$

$$-10k = \ln .65$$

$$k = -.1 \ln .65 = .043$$

$$T_{\frac{1}{2}} = \frac{\ln 2}{k} = 16.12 \qquad \text{days}$$

4.

$$f'(x) = -2 \ f(x) + 4 = -2(f(x) - 2)$$

So,

$$f(x) - 2 = Ce^{-2t}$$

Using $f(0) = -3$, we find $C = -5$ and so,

$$f(x) = 2 - 5e^{-2t}$$

5. To find the fourth order Taylor polynomial of $f(x) = \sin x$ centered at $x = \frac{\pi}{2}$ we compute:

$$f(\pi/2) = \sin\frac{\pi}{2} = 1$$

$$f'(\pi/2) = \cos\frac{\pi}{2} = 0$$

$$f''(\pi/2)\sin\frac{\pi}{2} = -1$$

and so on. Hence

$$P_4 = 1 - \frac{(x-\frac{\pi}{2})^2}{2} + \frac{(x-\frac{\pi}{2})^4}{4!}$$

Part II

6. a.

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + R_n$$

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where

$$|R_n(x)| \leq 3 \cdot \frac{x^{n+1}}{(n+1)!}$$

b. To make the error less than 0.001, we choose n so that $(x = .1)$

$$3\frac{(.1)^{n+1}}{(n+1)!} < .001$$

$$n = 2 \quad , \quad |R_2| < 3(.1)^3/3! = .001/2 = .0005$$

Hence we can estimate

$$e^{.1} \approx 1 + (.1) + (.1)^2/2 = 1 + .1 + .005 = 1.105$$

(The precise answer is 1.1051709...)

7a.

$$\begin{aligned} \int \frac{dy}{y(1-y)} &= \int (\frac{1}{y} + \frac{1}{1-y})\ dy \\ &= \ln |\frac{y(t)}{1-y(t)}| \end{aligned}$$

Hence,

$$\frac{y(t)}{1-y(t)} = Ce^{.1t}$$

Using $y(0) = .2$ we find $C = \frac{.2}{.8} = .25$.
Then

$$y(t) = \frac{.25e^{.1t}}{1 + .25e^{.1t}} = 1 - \frac{1}{1 + .25e^{.1t}}$$

b. $y(t) = .5$ exactly when $e^{.1t} = .5$ which gives the usual "half-life" time:

$$T = \frac{\ln 2}{.1} = 6.93$$