FINAL PRACTICE EXAM III

YI LI

1. Let $f(x,y) = e^{-xy}$ with constraint function

$$x^2 + 4y^2 = 1.$$

Using Lagrange multipliers to find all extrema.

Define $g(x,y) := x^2 + 4y^2 - 1$. Compute

$$\nabla f(x,y) = \begin{bmatrix} -y \\ -x \end{bmatrix} e^{-xy}, \quad \nabla g(x,y) = \begin{bmatrix} 2x \\ 8y \end{bmatrix}.$$

From the equation $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$, we get

$$\begin{bmatrix} -y_0 \\ -x_0 \end{bmatrix} e^{-x_0 y_0} = \lambda \begin{bmatrix} 2x_0 \\ 8y_0 \end{bmatrix}.$$

Hence

$$-y_0 e^{-x_0 y_0} = 2\lambda x_0, \quad -x_0 e^{-x_0 y_0} = 8\lambda y_0, \quad x_0^2 + 4y_0^2 = 1.$$

Thus

$$(x_0, y_0) = (1/\sqrt{2}, 1/2\sqrt{2}), (1/\sqrt{2}, -1/2\sqrt{2}), (-1/\sqrt{2}, 1/2\sqrt{2}), (-1/\sqrt{2}, -1/2\sqrt{2}).$$

2. Consider the system of linear equations

$$\begin{aligned} 3x + 5y - z &= 10 \\ 2x - y + 3z &= 9 \\ 4x + 2y - 3z &= -1 \end{aligned}$$

Find the augmented matrix of the above system and use it to solve the system.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 5 & -1 & 10 \\ 2 & -1 & 3 & 9 \\ 4 & 2 & -3 & -1 \end{array}\right]$$

Then

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left[\begin{array}{ccc|c} 3 & 5 & -1 & 10 \\ 2 & -1 & 3 & 9 \\ 4 & 2 & -3 & -1 \end{array}\right] \xrightarrow[R_1 - \frac{3}{4}R_3]{R_1 - \frac{3}{2}R_2} \begin{matrix} R_4 \\ R_5 \\ R_6 \end{matrix} \left[\begin{array}{ccc|c} 3 & 5 & -1 & 10 \\ 0 & \frac{13}{2} & -\frac{11}{2} & -\frac{7}{2} \\ 0 & \frac{7}{2} & \frac{5}{4} & \frac{43}{4} \end{array}\right]$$

$$\xrightarrow{R_5 - \frac{13}{7}R_6} \begin{matrix} R_7 \\ R_8 \\ R_9 \end{matrix} \left[\begin{array}{ccc|c} 3 & 5 & -1 & 10 \\ 0 & \frac{13}{2} & -\frac{11}{2} & -\frac{7}{2} \\ 0 & 0 & \frac{-219}{28} & \frac{-657}{28} \end{array}\right]$$

Therefore, $z = 3$, $y = 2$, and $x = 1$.

3. Assume that

$$f(x,y) = \begin{cases} 0, & \text{if } xy \neq 0, \\ 1, & \text{if } xy = 0. \end{cases}$$

(a) Show that $f_x(0,0)$ and $f_y(0,0)$ exist.
(b) $f(x,y)$ is not differentiable at $(0,0)$.

(a)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Similarly, we have $f_y(0,0) = 0$.

(b) To prove $f(x,y)$ is not differentiable at $(0,0)$, we suffice to show that $f(x,y)$ is discontinuous at $(0,0)$. Consider two special paths $C_1 : y = 0$ and $C_2 : y = x$. Then

$$\lim_{(x,y)\to(0,0) \text{ along with } C_1} f(x,y) = 1$$

and

$$\lim_{(x,y)\to(0,0) \text{ along with } C_2} f(x,y) = \lim_{x\to 0} f(x,x) = 0.$$

Therefore $f(x,y)$ is not continuous at $(0,0)$.

4. Compute

$$\int_0^1 \frac{dx}{(x-1)^{2/3}}.$$

By definition,

$$\begin{aligned} \int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{c\to 1-} \int_0^c \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1-} 3(x-1)^{1/3}\Big|_0^c \\ &= \lim_{c\to 1-} \left[3(c-1)^{1/3} - 3(0-1)^{1/3}\right] = 0+3 = 3. \end{aligned}$$

5. Find the absolute maxima and minima of $f(x,y) = x^2 + y^2 + x + 2y$ on the disk $D = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \leq 4\}$.

Compute

$$\nabla f(x,y) = \begin{bmatrix} 2x+1 \\ 2y+2 \end{bmatrix}, \quad \textbf{Hess}(f)(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The critical point inside of D is $(-1/2, -1)$ at which the function $f(x,y)$ has a local minimum $f(-1/2,-1) = -5/4$.

On the boundary, we wish to maximize/minimize $f(x,y)$ with the constraint $g(x,y) := x^2 + y^2 - 4 = 0$. Since

$$\nabla g(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix},$$

and the equation $\nabla f(x,y) = \lambda g(x,y)$ we obtain

$$2x+1 = 2\lambda x, \quad 2y+2 = 2\lambda y, \quad x^2+y^2 = 4.$$

A simple computation shows

$$(x,y) = (2/\sqrt{5}, 4/\sqrt{5}),\ (-2/\sqrt{5}, -4/\sqrt{5}).$$

The global minimum is $-5/4$ and the global maximum is $f(2/\sqrt{5}, 4/\sqrt{5}) = 4+2\sqrt{5}$.

6. Use the partial-fraction method to solve

$$\frac{dy}{dx} = 2y(3-y)$$

with $y(1) = 5$.

Compute

$$-2dx = \frac{dy}{y(y-3)} = \frac{1}{3}\left(\frac{1}{y-3} - \frac{1}{y}\right) dy.$$

Then

$$\ln\left|\frac{y-3}{y}\right| = -6x + C_1 \Longrightarrow \frac{y-3}{y} = Ce^{-6x}.$$

Since $y(1) = 5$, it follows that $C = \frac{2}{5}e^6$ and then

$$y = \frac{3}{1 - \frac{2}{5}e^{6-6x}}.$$

7. Find the local extrema of

$$f(x,y) = 2x^2 - xy + y^4, \quad (x,y) \in \mathbf{R}^2.$$

Compute

$$\nabla f(x,y) = \begin{bmatrix} 4x - y \\ -x + 4y^3 \end{bmatrix}, \quad \mathbf{Hess}(f)(x,y) = \begin{bmatrix} 4 & -1 \\ -1 & 12y^2 \end{bmatrix}$$

Then all critical points are $(0,0), (1/16, 1/4), (-1/16, -1/4)$.

Then $f(x,y)$ has a local maximum at $(0,0)$ while has local maxima at $(1/16, 1/4)$ and $(-1/16, -1/4)$.

8. Let

$$A = \begin{bmatrix} 2 & 4 \\ -2 & -3 \end{bmatrix}$$

Without explicitly computing the eigenvalues of A, decide whether the real parts of both eigenvalues are negative.

Since $\det A = 2$ and $\operatorname{tr} A = -1$, the real parts of both eigenvalues are negative.

9. Solve the given initial-value problem

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

with $x_1(0) = -3$ and $x_2(0) = 1$.

Let

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}.$$

From $\det A = 0$ and $\operatorname{tr} A = 5$, we get $0 = \lambda^2 - 5\lambda$, where λ is an eigenvalue of A. Hence $\lambda_1 = 0$ and $\lambda_2 = 5$.

For $\lambda_1 = 0$, we have

$$\mathbf{0} = (A - \lambda_1 I_2)\mathbf{u} \Longrightarrow u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 5$, we have

$$\mathbf{0} = (A - \lambda_2 I_2)\mathbf{v} \Longrightarrow v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

When $x_1(0) = -3$ and $x_1(0) = 1$, we get $c_1 = 1$ and $c_2 = 0$. Hence $\mathbf{x}(t) = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

10. Suppose that

$$\frac{dy}{dx} = y(2-y)(y-3).$$

(a) Find the equilibria of this differential equation.

(b) Compute the eigenvalues associated with each equilibrium and discuss the stability of the equilibria.

Let $g(y) := y(2-y)(y-3)$. Then $g'(y) = -3y^2 + 10y - 6$.

(a) All equilibria are $0, 2$, and 3.

(b) Since $g'(0) = -6$, $g'(2) = 2$, and $g'(3) = -3$, we see that $0, 3$ are locally stable, and 2 is unstable.

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