FINAL PRACTICE EXAM I (SOLUTION)

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1. Determine if the following improper integral converges or diverges. If the integral is convergent compute its value.

$$\int_0^\infty xe^{-x}\,dx.$$

By integration by parts, we have

$$\begin{aligned}\int_0^\infty xe^{-x}\,dx &= -\int_0^\infty x\,d\left(e^{-x}\right) = -\left(xe^{-x}\Big|_0^\infty - \int_0^\infty e^{-x}\,dx\right)\\ &= -\left(0-\int_0^\infty e^{-x}\,dx\right) = \int_0^\infty e^{-x}\,dx\\ &= -e^{-x}\Big|_0^\infty = -(0-1) = 1.\end{aligned}$$

2. Let

$$f(x,y)=\begin{cases}\frac{3x^2y^2}{x^3+y^6}, & \text{if } (x,y)\neq(0,0),\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

(a) Does the $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?
(b) Is $f(x,y)$ continuous at $(0,0)$?

(a) For the line $C_1 : y = mx$, we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)\text{ along with } C_1} f(x,y) &= \lim_{x\to0} f(x,mx) = \lim_{x\to0}\frac{3m^2x^4}{x^3+m^6x^6}\\ &= \lim_{x\to0}\frac{3m^2x}{1+m^6x^3} = 0.\end{aligned}$$

For the line $C_2 : x = y^2$, we have

$$\lim_{(x,y)\to(0,0)\text{ along with } C_2} f(x,y) = \lim_{y\to0} f(y^2,y) = \lim_{y\to\infty}\frac{3y^6}{2y^6} = \frac{3}{2}.$$

Hence, the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.
(b) By part (a), it immediately follows that $f(x,y)$ is discontinuous at $(0,0)$.

3. Solve the following first order separable initial value problem

$$\frac{dy}{dx} = (y-1)(y-2)$$

with $y(0) = 0$.

Write the differential equation as

$$\frac{dy}{(y-1)(y-2)} = dx.$$

The rational function $1/(y-1)(y-2)$ has the following partial fraction:

$$\frac{1}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2}$$

for some constants A and B. Since

$$\frac{A}{y-1} + \frac{B}{y-2} = \frac{A(y-2)+B(y-1)}{(y-1)(y-2)} = \frac{(A+B)y-(2A+B)}{(y-1)(y-2)}$$

it follows that

$$A+B=0, \quad 2A+B=-1.$$

Solving the above two linear equations yields $A=-1$ and $B=1$. Hence

$$\frac{1}{(y-1)(y-2)} = \frac{-1}{y-1} + \frac{1}{y-2} = \frac{1}{y-2} - \frac{1}{y-1}.$$

Plugging this decomposition into above, we arrive at

$$\left(\frac{1}{y-2} - \frac{1}{y-1}\right) dy = dx \Longrightarrow \ln\left|\frac{y-2}{y-1}\right| = x + C_1$$

for some constant. Thus

$$\frac{y-2}{y-1} = Ce^x \quad (C = \pm e^{C_1}) \Longrightarrow y = \frac{2-Ce^x}{1-Ce^x}.$$

Using $y(0)=0$, we get $0 = \frac{2-C}{1-C}$ and then $C=2$. Hence

$$y = \frac{2-2e^x}{1-2e^x}.$$

4. Consider the system of linear equations

$$\begin{aligned} x_1 - x_2 &= 0 \\ 3x_1 + x_2 - x_3 &= 11 \\ 2x_1 + x_2 + 2x_3 &= 11 \end{aligned}$$

Find the augmented matrix of the above system and use it to solve the system.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 3 & 1 & -1 & 11 \\ 2 & 1 & 2 & 11 \end{array}\right]$$

Then

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 3 & 1 & -1 & 11 \\ 2 & 1 & 2 & 11 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \begin{array}{c} R_4 \\ R_5 \\ R_6 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 4 & -1 & 11 \\ 0 & 3 & 2 & 11 \end{array}\right]$$

$$\xrightarrow{-\frac{3}{4}R_5+R_6} \begin{array}{c} R_7 \\ R_8 \\ R_9 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 4 & -1 & 11 \\ 0 & 0 & \frac{11}{4} & \frac{11}{4} \end{array}\right]$$

Therefore, $z = 1$, $y = 3$, and $x = 3$.

5. Consider $f(x, y) = 3xy - x^3 - y^3$.
(a) Locate all critical points of $f(x, y)$.
(b) Classify the critical points of $f(x, y)$ (i.e., determine if they are local maximum/local minimum or saddle point).
(c) Does f have a global maximum or minimum on $\mathbf{R}^2$? Briefly explain!

Since $f(x, y)$ is differentiable at any points of $\mathbf{R}^2$, all critical points must satisfy

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \nabla f(x, y) = \begin{bmatrix} 3y - 3x^2 \\ 3x - 3y^2 \end{bmatrix}$$

(a) If (x_0, y_0) is a critical point, then

$$y_0 - x_0^2 = 0, \quad x_0 - y_0^2 = 0.$$

Thus $(x_0, y_0) = (0, 0)$ or $(1, 1)$.
(b) The Hessian matrix of f is

$$\mathbf{Hess}(f)(x, y) = \begin{bmatrix} -6x & 3 \\ 3 & -6y \end{bmatrix}$$

For $(0, 0)$, its Hessian matrix has the form

$$H = \mathbf{Hess}(f)(0, 0) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \quad \det H = -9 < 0;$$

this critical point is a saddle point.
For $(1, 1)$, its Hessian matrix has the form

$$H = \mathbf{Hess}(f)(1, 1) = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}, \quad \det H = 27 > 0, \quad \operatorname{tr} H = -12 < 0;$$

the function $f(x, y)$ has a local maximum at this critical point.
(c) Consider a line $y = ax$. Then

$$f(x, y) = f(x, ax) = 3ax^2 - x^3 - a^3x^3.$$

Choose $a = 0$, we have

$$f(x, 0) = -x^3 \to -\infty \quad \text{as } x \to \infty$$

and

$$f(x, 0) = -x^3 \to +\infty \quad \text{as } x \to -\infty.$$

This means that f has no global extrema.

6. Consider the following system of differential equations

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Solve the following initial value problem with $x_1(0) = 5$ and $x_2(0) = 3$.

Let

$$A = \begin{bmatrix} -5 & -2 \\ 6 & 3 \end{bmatrix}$$

From $\det A = -3$ and $\operatorname{tr} A = -2$, we get

$$0 = \lambda^2 - \lambda \operatorname{tr} A + \det A = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1).$$

The two eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -3$.

If $\mathbf{u}$ is an eigenvector of λ_1, then

$$0 = (A - \lambda_1 I_2)\mathbf{u} = \left(\begin{bmatrix} -5 & -2 \\ 6 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and $u_2 = -3u_1$. Hence

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ -3u_1 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

If $\mathbf{v}$ is an eigenvector of λ_2, then

$$0 = (A - \lambda_1 I_2)\mathbf{u} = \left(\begin{bmatrix} -5 & -2 \\ 6 & 3 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

and $v_2 = -v_1$. Hence

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The general solution now is given by

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 e^{-3t} \\ -3c_1 e^t - c_2 e^{-3t} \end{bmatrix}.$$

From $x_1(0) = 5$ and $x_2(0) = 3$, we find that

$$c_1 + c_2 = 5, \quad -3c_1 - c_2 = 3.$$

Solving those linear equations gives us $c_1 = -4$ and $c_2 = 9$. Consequently

$$\mathbf{x}(t) = \begin{bmatrix} -4e^t + 9e^{-3t} \\ 12e^t - 9e^{-3t} \end{bmatrix}.$$

7. Find the absolute maxima and minima of $f(x, y) = x^2 + y^2 - 2x + 4$ on the disk $D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \le 4\}$.

Compute

$$\nabla f(x, y) = \begin{bmatrix} 2x - 2 \\ 2y \end{bmatrix}, \quad \mathbf{Hess}(f)(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

so that the function $f(x, y)$ has the only critical point $(1, 0)$ inside of D. and f has a local minimum $f(1, 0) = 3$ at this point.

Any point of the boundary of D can be written, in terms of polar coordinates, as

$$x = 2\cos\theta, \quad y = 2\sin\theta, \quad \theta \in [0, 2\pi).$$

Hence

$$f(x, y) = f(2\cos\theta, 2\sin\theta) = 4 - 4\cos\theta + 4 = 4(2 - \cos\theta), \quad \theta \in [0, 2\pi),$$

on the boundary ∂D. When $\cos\theta = 1$ (or $\theta = 0$), f has the minimum 4, thus f has the minimum 4 at the point $(2, 0)$; when $\cos\theta = -1$ (or $\theta = \pi$), f has the maximum 12, thus f has the maximum 12 at the point $(-2, 0)$.

Therefore the absolute maxima and minima are 12 and 3 respectively.

8. Let $f(x,y) = \sqrt{4x^2+y^2}$ be a function of two variables.

(a) Compute the directional derivative of the function $f(x,y)$ at the point $(-2,4)$ in the direction of $\mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

(b) Find the angle between the vectors $\nabla f(-2,4)$ and $\mathbf{v}$.

The gradient vector of f is

$$\nabla f(x,y) = \begin{bmatrix} \frac{4x}{\sqrt{4x^2+y^2}} \\ \frac{y}{\sqrt{4x^2+y^2}} \end{bmatrix}$$

(a) The gradient of f at $(-2,4)$ is

$$\nabla f(-2,4) = \begin{bmatrix} \frac{-8}{4\sqrt{2}} \\ \frac{4}{4\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

The normalization of $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$ is

$$\mathbf{u} = \frac{1}{\left\| \begin{bmatrix} -3 \\ -1 \end{bmatrix} \right\|} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

so that the directional derivative is equal to

$$D_{\mathbf{u}} f(-2.4) = \nabla f(-2.4) \cdot \mathbf{u} = \begin{bmatrix} \frac{-2}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \frac{8}{\sqrt{20}} = \frac{4}{\sqrt{5}}.$$

(b) Let $\mathbf{w} = \nabla f(-2,4)$. Recall the formula

$$\cos\theta = \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{w}||\mathbf{v}|}$$

where θ is the angle between $\mathbf{w}$ and $\mathbf{v}$. The length of $\mathbf{w}$ is

$$|\mathbf{w}| = \sqrt{\frac{4}{2} + \frac{1}{2}} = \frac{\sqrt{5}}{\sqrt{2}}.$$

Hence

$$\cos\theta = \frac{\frac{6}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{\frac{\sqrt{5}}{\sqrt{2}}\sqrt{10}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Thus the angel is $45°$.

9. Suppose you wish to enclose a rectangle plot. You have 1600 ft of fencing. Using the material, what are the dimensions of the plot that will have the largest area?

We wish to maximize

$$A = xy$$

subject to the constraint $2x + 2y = 1600$. Consider

$$f(x,y) = xy, \quad g(x,y) = 2x + 2y - 1600 = 0.$$

From

$$\nabla f(x, y) = \begin{bmatrix} y \\ x \end{bmatrix}, \quad \nabla g(x, y) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

and the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$, we have

$$y = 2\lambda, \quad x = 2\lambda, \quad x + y = 800,$$

from which we get $\lambda = 200$ and $x = y = 400$, with $f(400, 400) = 160,000$.

We now look at the boundary. By the physical reason, $x, y > 0$, so that we need only to consider the line segment $x + y = 800$ with $0 < x < 800$. On this line segment, we have

$$f(x, y) = f(x, 800 - x) = x(800 - x) = -x^2 + 800x, \quad 0 < x < 800.$$

The maximum value takes at $(400, 400)$. Consequently, the largest area is $f(400, 400) = 160,000$.

10. Suppose that

$$\frac{dy}{dx} = y(2 - y).$$

(a) Find the equilibria of this differential equation.

(b) Compute the eigenvalues associated with each equilibrium and discuss the stability of the equilibria.

Let $g(y) = y(2 - y)$. Then $g'(y) = 2 - 2y$.

(a) The equilibria are $y = 0$ and $y = 2$.

(b) Since $g'(0) = 2 > 0$ and $g'(2) = -2$, the equilibrium 0 is unstable while 2 is locally stable.

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