Math 107 Exam 2 Solutions

1a. $y = \cosh x = \frac{e^x+e^{-x}}{2}$ from $x = 0$ to $x = 1$. $y' = \sinh x = \frac{e^x-e^{-x}}{2}$ and $1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$. So

$$L = \int_0^1 \cosh x \; dx = \sinh x|_0^1 = \sinh 1 \ .$$

b. Revolving $y = \cosh x$ about the x axis from $x = 0$ to $x = 1$, the surface area is given by

$$\begin{aligned} S &= 2\pi \int_0^1 \cosh x \cdot \cosh x \; dx = 2\pi \int_0^1 \cosh^2 \; dx = 2\pi \int_0^1 (\frac{e^x + e^{-x}}{2})^2 \; dx \\ &= \frac{2\pi}{4} \int_0^1 (e^{2x} + 2 + e^{-2x}) \; dx = \frac{\pi}{2}[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}]_0^1 = \frac{\pi}{2}[\sinh 2 + 2] \end{aligned}$$

2. $p(x) = cx^{-3}$ on $[1, \infty)$.
a. $\int_1^\infty x^{-3} \; dx = \frac{1}{2}$ so $c = 2$.

b. $\mu = \int_1^\infty x \cdot 2x^{-3} \; dx = 2 \int_0^\infty x^{-2} \; dx = 2$.

c. $F(x) = \int_1^x 2t^{-3} \; dt = 1 - x^{-2}$. To find the median we solve $F(x) = \frac{1}{2}$, i.e. $1 - x^{-2} = \frac{1}{2}$. Hence $x^{-2} = \frac{1}{2}$ so $x = \sqrt{2}$ is the median.

3.

$$\begin{aligned} \int_1^\infty \frac{\ln x}{x^2} \; dx &= \lim_{L\to\infty} \int_0^L \frac{\ln x}{x^2} \; dx = \lim_{L\to\infty} \{-\frac{\ln x}{x} + \int_1^L \frac{1}{x^2} \; dx\} \\ &= \lim_{L\to\infty} \{-\frac{\ln x}{x} - \frac{1}{x}\}_1^L = 1 \end{aligned}$$

4b. $y = \sin x$, $x_0 = 0$, $x_1 = \frac{\pi}{6}$, $x_2 = \frac{\pi}{3}$, $x_3 = \frac{\pi}{2}$
$x_4 = \frac{2\pi}{3}$, $x_5 = \frac{5\pi}{6}$, $x_6 = \pi$.

$y_0 = 0$, $y_1 = \frac{1}{2}$, $y_2 = \frac{\sqrt{3}}{2}$, $y_3 = 1$, $y_4 = \frac{\sqrt{3}}{2}$, $y_5 = \frac{1}{2}$, $y_6 = 0$. Hence,

$$S_6 = \frac{\pi}{18}\{0 + 4 \cdot \frac{1}{2} + 2\frac{\sqrt{3}}{2} + 4 \cdot 1 + 2\frac{\sqrt{3}}{2} + 4 \cdot \frac{1}{2} + 0\} = \frac{\pi}{18}(8 + 2\sqrt{3}) = 2.0084$$

5. $k = 0.05$, $p(x) = 0.05e^{-0.05x}$, $F(x) = 1 - e^{-0.05x}$. Hence the probability that a random S&L survives more than 40 years is $1 - F(40) = e^{-2}$.

6.$\mu = 30,000$, $\sigma = 2000$. From the Z-table below, the top 10% corresponds to 1.28 standard deviations above the mean (i.e $Z = 1.28$). We convert back to the given normal distribution using $X = \sigma Z + \mu$. So,

$$X = (1.28)(2000) + 30,000 = 32,560 \text{ miles.}$$

Z	Area from 0 to Z
1.25	0.3944
1.26	0.3962
1.27	0.3980
1.28	0.3997
1.29	0.4015
1.30	0.4032