Name: ____ **Section Number:**___________

110.107 CALCULUS II (Biology & Social Sciences)
FALL 2010
MIDTERM EXAMINATION
December 1, 2010

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		35
2		30
3		35
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ____ **Date:**___________

Question 1. [35 points] Let $G(x,y) = x^2 + xy^2 - \frac{y^2}{2}$, and let $\mathbf{p} = (-1,1) \in \mathbb{R}^2$. Do the following:

(a) Calculate the equation of the plane tangent to the graph of G at the point $\mathbf{p}$.

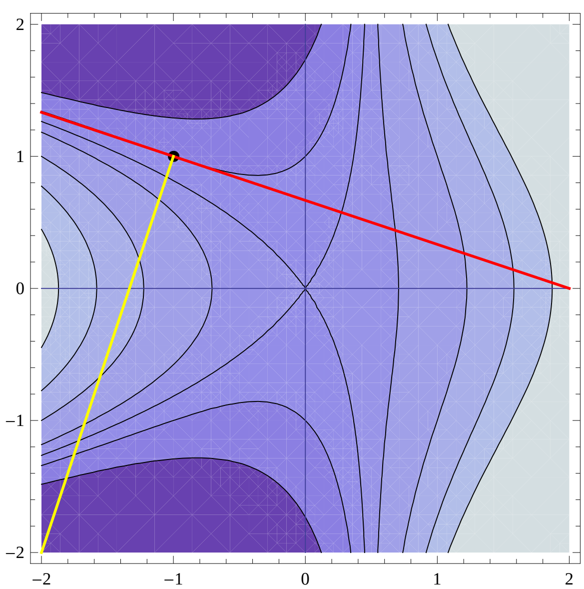
Solution: The equation of the plane in $\mathbb{R}^3$ written as a function of two of the variables can be written $z - z_0 = A(x - x_0) + B(y - y_0)$, where x_0, y_0, z_0, A, and B are constants. If the plane is tangent to the graph of a function $G(x,y)$ at the point (x_0, y_0), then the other constants are $z_0 = G(X_0, y_0)$ and $A = \frac{\partial G}{\partial x}(x_0, y_0)$, and $B = \frac{\partial G}{\partial y}(x_0, y_0)$. In our case, $G(x,y) = x^2 + xy^2 - \frac{y^2}{2}$, so

$$\begin{aligned} z_0 &= G(-1,1) = (-1)^2 + (-1)(1)^2 - \frac{(1)^2}{2} = -\frac{1}{2}, \\ A &= \frac{\partial G}{\partial x}(-1,1) = \left(2x + y^2\right)\Big|_{x=-1,y=1} = -1, \\ B &= \frac{\partial G}{\partial y}(-1,1) = (2xy - y)\Big|_{x=-1,y=1} = -3. \end{aligned}$$

Thus the equation of the plane here is $z - \left(-\frac{1}{2}\right) = (-1)\left(x - (-1)\right) + (-3)\left(y - 1\right)$, or

$$\boxed{z + \frac{1}{2} = -(x+1) - 3(y-1).}$$

(b) The point $\mathbf{p}$ is graphed on the contour plot shown here. Determine which level set $\mathbf{p}$ lives on, and calculate $\frac{dy}{dx}$ at $\mathbf{p}$. Draw on the contour plot the line through $\mathbf{p}$ with this slope.



Solution: By the previous problem, $G(-1,1) = (-1)^2 + (-1)(1)^2 - \frac{(1)^2}{2} = -\frac{1}{2}$. Hence the point $\mathbf{p} = (-1,1)$ lives on the $-\frac{1}{2}$-level set. This level-set, is simply the set of solutions in $\mathbb{R}^2$ to the equation $x^2 + xy^2 - \frac{y^2}{2} = \frac{1}{2}$. Hence finding how y varies as we vary x on this level set at the point $\mathbf{p}$, is straightforward: Differentiate implicitly. Here, differentiate the entire equation $\frac{d}{dx}\left(x^2 + xy^2 - \frac{y^2}{2} = -\frac{1}{2}\right)$. We get

$$2x + y^2 + 2xy\frac{dy}{dx} - y\frac{dy}{dx} = 0.$$

Solving for the unknown derivative, we get

$$\frac{dy}{dx} = \frac{-2x - y^2}{2xy - y}.$$

Note that this is the same as if we followed the multi-variable version of implicit differentiation, by first computing $G_x = \frac{\partial G}{\partial x} = 2x + y^2$, and $G_y = \frac{\partial G}{\partial y} = 2xy - y$, and then using the formula $\frac{dx}{dy} = \frac{-G_x}{G_y}$. Now evaluating at $\mathbf{p}$, we get

$$\boxed{\left.\frac{dy}{dx}\right|_{\mathbf{p}} = \left.\left(\frac{-2x - y^2}{2xy - y}\right)\right|_{x=-1,y=1} = -\frac{1}{3}.}$$

The line is drawn on the contour plot in RED.

(c) Calculate the gradient of G at the point $\mathbf{p}$ (that is, compute $\nabla G(-1, 1)$). Draw this vector on the contour plot above.

Solution: The gradient vector is just the derivative of G, written as a 2-vector. Here

$$\nabla G(-1, 1) = \left[\begin{array}{c} \frac{\partial G}{\partial x} \\ \frac{\partial G}{\partial y} \end{array}\right]_{x=-1,y=1} = \left[\begin{array}{c} 2x + y^2 \\ 2xy - y \end{array}\right]_{x=-1,y=1} = \left[\begin{array}{c} -1 \\ -3 \end{array}\right].$$

On the contour plot, I show the gradient vector in YELLOW. I couldn't draw the vector head down at the corner of the graph, but the base of the gradient vector should be at $\mathbf{p}$.

(d) Find a critical point of $G(x, y)$ on the domain $\mathbb{R}^2$ and determine whether it is a local maximum, local minimum, or neither (Hint: The Hessian will help here.)

Solution: Note that G is a polynomial function, and hence differentiable everywhere. So the only critical points will be where the derivative matrix is the zero matrix. Since the gradient vector carries the same information, we can say that the critical points are the places where the gradient vector is the 0-vector. So set the general gradient vector, as a vector of functions, to the 0-vector and solve: You get the two equations

$$2x + y^2 = 0$$
$$2xy - y = 0.$$

The latter equation $(2x-1)y = 0$ is solved when either $y = 0$ or $x = \frac{1}{2}$. But for $x = \frac{1}{2}$, the first equation has no solutions (try solving $1 + y^2 = 0$). So we ignore $x = \frac{1}{2}$ as part of a critical point. Letting $y = 0$, we find by substitution in to the first equation that also $x = 0$. Hence the origin is the ONLY critical point. As for the type of critical point, we can use the Hessian to compute this:

$$Hess(G)(0, 0) = \left[\begin{array}{cc} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{array}\right]_{x=0,y=0} = \left[\begin{array}{cc} 2 & 2y \\ 2y & 2x - 1 \end{array}\right]_{x=0,y=0} = \left[\begin{array}{cc} 2 & 0 \\ 0 & -1 \end{array}\right].$$

Here $\det Hess(G)(0, 0) = -2 < 0$. By the Second Derivative Test for a function of more than one independent variable, the means that the critical point is a saddle, and neither a local minimum nor a local maximum.

Question 2. [30 points] Given the system

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = 4x - 2y,$$

do the following:

(a) Solve the system for the particular solution that passes through the point $(x, y) = (1, 0)$.

Solution: Rewritten as a matrix equation,

$$\dot{\vec{x}} = A\vec{x}, \qquad \text{where } \vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \text{ and } A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}.$$

Given this system, it is the properties of the matrix A that allow us to write out the solutions: Namely, the general solution to a system like this is $\vec{x}(t) = c_1\vec{u}_1e^{\lambda_1 t} + c_2\vec{u}_2e^{\lambda_2 t}$, where $\lambda_1.\lambda_2$ are the two eigenvalues of A, with corresponding eigenvectors $\vec{u_1}, \vec{u}_2$, at least when the two eigenvalues are real and distinct. Here, the eigenvalues of A satisfy the characteristic equation $\lambda^2 - (\text{tr } A)\lambda + \det A = 0$. For us, this means that the equation is $\lambda^2 + 1 - 6 = 0$. This is solved by $\lambda = 2$ and $\lambda = -3$. For $\lambda = 2$, the eigenvector equation $A\vec{u} = 2\vec{u}$ leads to the system $x + y = 2x$ and $4x - 2y = 2y$. Remember these two equations are ALWAYS the same equation when finding the eigenvectors, and any vector $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ that satisfies either equation one works. The first equation leads directly to $x = y$. Choose $x = 1$, so that $y = 1$, and an eigenvector for $\lambda_1 = 2$ is $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If we do the same thing for $\lambda_2 = -3$, we will get the single equation $x + y = -3x$, and if we choose $x = 1$, we get $y = -4$, and for $\lambda_2 = 03$, we get $\vec{u}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$. Hence the general solution to this system is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}.$$

For our particular solution, we have a starting point $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Throw that into the general solution evaluated at $t = 0$, to get the values of the two unknown constants c_1 and c_2 that correspond to the solution that passes through the point $(1, 0)$. Here

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^0 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

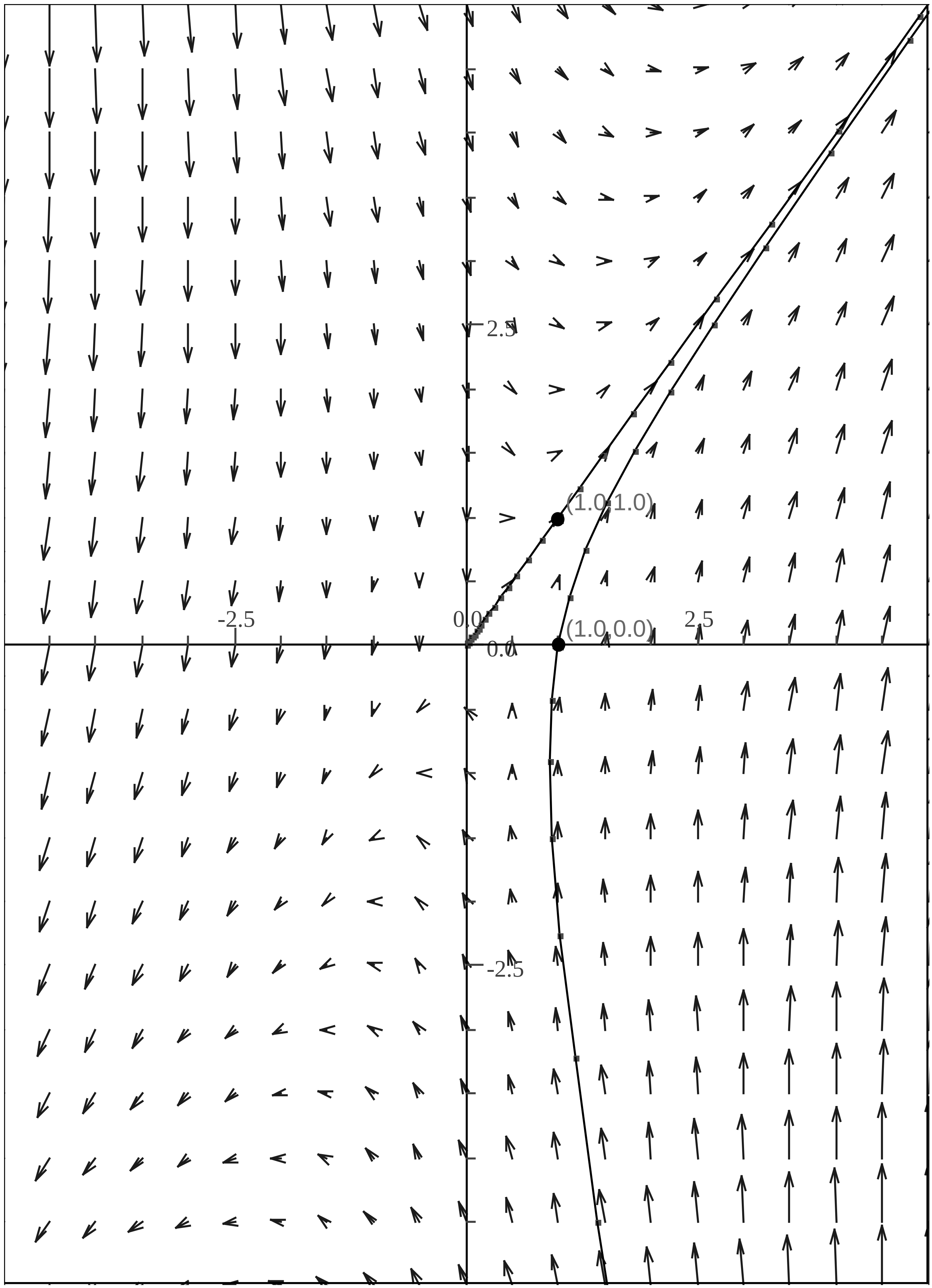
This leads to the two equations $1 = c_1 + c_2$ and $0 = c_1 - 4c_2$. Solving these two leads to $c_1 = \frac{4}{5}$ and $c_2 = \frac{1}{5}$. Hence our particular solution is

$$\boxed{\vec{x}(t) = \frac{4}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \frac{1}{5} \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}.}$$

(b) Find all equilibrium solutions and determine their stability.

Solution: The only equilibrium of a linear system where the matrix A has non-zero determinant (like this one) is the origin. And since the two eigenvalues here are real, distinct, non-zero, and of different signs, the origin is a saddle and unstable.

(c) On the next page, draw the solution passing through the point $(x, y) = (1, 0)$ on the direction field for all $t \in \mathbb{R}$. Also draw the solution passing through the point $(x, y) = (1, 1)$ for all $t \in \mathbb{R}$. Note that these two points are marked.



2.5
(1.0,1.0)
-2.5
0.0
2.5
(1.0,0.0)
0.0
-2.5

Question 3. [35 points] Consider the following experiment: Flip three coins in succession. The first two are fair, but the last has a 60% chance of landing heads. Do the following:

(a) Write out the sample space for this experiment and assign a probability to it.

Solution: Here the sample space is just the same as if the three coins were fair:

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

However, assigning a probability will be a little bit different since the last coin is not fair. Since each coin is independent from the others, we can calculate their probabilities by using the probability of each flipped coin separately and multiplying the probabilities of the combined flips together. This works because each coin flip is independent from the other two. For example,

$$\begin{aligned} P(HHH) &= P(\{\text{first is heads}\} \cap \{\text{second is heads}\} \cap \{\text{third is heads}\}) \\ &= P(\{\text{first is heads}\}) \cdot P(\{\text{second is heads}\}) \cdot P(\{\text{third is heads}\}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{6}{10} = \frac{6}{40}. \end{aligned}$$

Doing the rest this way, we find that a good probability for this sample space will assign the respective probabilities to the outcomes $\left\{\frac{6}{40}, \frac{4}{40}, \frac{6}{40}, \frac{4}{40}, \frac{6}{40}, \frac{4}{40}\right\}$. That is, any outcome with a heads on the last coin will have a probability of $\frac{6}{40}$, and any outcome with a tails on the last coin will have a probability of $\frac{4}{40}$.

(b) Define X to be a discrete random variable corresponding to the number of heads. Write out the probability mass function $p(X)$.

Solution: We can use the sample space Ω directly to calculate the valid values for the random variable X by simply counting heads in the outcomes. We get

$$X = \{0, 1, 2, 3\}.$$

And we can assign probabilities to the random variable X (this is what the probability mass function does) by simply adding up the probabilities of each outcome that corresponds to a particular value of X. For example,

$$p(2) = P(X = 2) = P(HHT) + P(HTH) + P(THH) = \frac{4}{40} + \frac{6}{40} + \frac{6}{40} = \frac{16}{40}.$$

Done this way, we get

$$p(0) = \frac{4}{40}, \quad p(1) = \frac{14}{40}, \quad p(2) = \frac{16}{40}, \quad p(3) = \frac{6}{40}.$$

(c) let $A = \{\text{at least two heads}\}$, $B = \{\text{at most two tails}\}$, $C = \{\text{exactly two heads}\}$, and $D = \{\text{the last coin is a heads}\}$.

- Find $P(A)$ and $P(B)$.
- determine whether A and B are independent events.
- Calculate $P(D \mid C)$.
- Find $E(X)$.

Solution: For these probabilities, one can use either the original sample space Ω with its probability, or the probability mass function $p(X)$ of the random variable X. For the probability of event A,

$$P(A) = P(HHH) + P(HHT) + P(HTH) + P(THH) = \frac{6+4+6+6}{40} = \frac{22}{40}.$$

But also, $P(A) = p(2) + p(3) = \frac{16+6}{40} = \frac{22}{40}$. For event B,

$$\begin{aligned} P(B) &= P(HHH) + P(HHT) + P(HTH) + P(THH) + P(HTT) + P(THT) + P(TTH) \\ &= \frac{6+4+6+6+4+4+6}{40} = \frac{36}{40}. \end{aligned}$$

Now to check the independence of events A and B, we first calculate $P(A \cap B)$. But notice that event $A \subset B$ (if something has at least two heads, it must also have at most ONE tail. hence it definitely will have at most two tails, right?). Hence it is the case that $A \cap B = A$, and hence $P(A \cap B) = P(A) = \frac{22}{40}$. The events are independent if $P(A \cap B) = P(A)P(B)$. But here

$$\frac{22}{40} = P(A \cap B) = P(A) \neq P(A)P(B) = \frac{22}{40}\frac{36}{40}.$$

Hence events A and B are not independent. And lastly, the expected value of X is simply the weighted sum of its values:

$$E(X) = \sum_{i=0}^{3} xP(X = x) = 0 \cdot \frac{4}{40} + 1 \cdot \frac{14}{40} + 2 \cdot \frac{16}{40} + 3 \cdot \frac{6}{40} = \frac{64}{40} = 1.6.$$

(If all three coins were fair, the expected value of X would have been 1.5. Just sayin'...).