Name: ____________________

110.107 CALCULUS II (Biology & Social Sciences)
FALL 2010
MIDTERM EXAMINATION
October 12, 2010

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		15
2		20
3		20
4		15
5		15
6		15
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ____________________ **Date:** __________

Question 1. [15 points] Let $A = \begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix}$. Do the following:

(a) Find $\det A$ and calculate A^{-1}.

Solution: Remember the determinant of a 2×2 matrix is the product of the main diagonal minus the product of the "other" diagonal. In this case,

$$\det A = 1 \cdot 3 - 3 \cdot 5 = 3 - 15 = -12.$$

The inverse has a form also. One switches the main diagonal elements, multiplies the "other" diagonal elements by -1, and then divides by the determinant. Here, we get

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} 3 & -5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{12} & \frac{5}{12} \\ \frac{3}{12} & -\frac{1}{12} \end{bmatrix}.$$

(b) Calculate the eigenvalues and corresponding eigenvectors of A

Solution: The eigenvalues are the constants λ in the eigenvalue/eigenvector equation $A\vec{x} = \lambda\vec{x}$. One can manipulate this equation into the form $(A - \lambda I_2)\,\vec{x} = 0$. What is in the parentheses here is a matrix, and since we are looking for non-trivial solutions $\vec{x}$ in this last equation, it will mean that the determinant of what is in the parentheses will have to be 0. This leads to the new equation $\det(A - \lambda I_2) = 0$, and is called the characteristic equation of A. It's solutions λ are the eigenvalues of A. Here

$$\begin{aligned} 0 = \det(A - \lambda I_2) &= \det\left(\begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1-\lambda & 5 \\ 3 & 3-\lambda \end{bmatrix}\right) = (1-\lambda)(3-\lambda) - 15 = \lambda^2 - 4\lambda - 12. \end{aligned}$$

Since this last equation factors to $(\lambda - 6)(\lambda + 2) = 0$, the two solutions (the two eigenvalues of A) are $\lambda = 6$ and $\lambda = -2$. For the eigenvectors for each λ, plug in the value for λ into the original equation $A\vec{x} = \lambda\vec{x}$, and solve for the unknown vector. For example, for $\lambda = 6$, we get $A\vec{x} = 6\vec{x}$, which leads to the system

$$\begin{aligned} x + 5y &= 6x \\ 3x + 3y &= 6y. \end{aligned}$$

This system will have tons of solutions (the eigenvector equation always does since any multiple of an eigenvector is also an eigenvector). So to solve, simply take either of the two equations and find values of x and y that work: Here $3x + 3y = 6y$ reduces to $3x - 3y = 0$ which is solved by any pair x and y when $x = y$. So choose $x = 1$. Then $y = 1$ and an eigenvector for $\lambda = 6$ if $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda = -2$, we will get a system with the equation $x + 5y = -2x$, which reduces to $3x = -5y$. Choose $x = 5$, $y = -3$, and an eigenvector for $\lambda = -2$ is $\vec{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

(c) Solve the matrix question $A\vec{x} = \vec{b}$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$.

Solution: One way to do this is to write out the system as equation and solve: $A\vec{x} = \vec{b}$ is

$$\begin{aligned} x + 5y &= 4 \\ 3x + 3y &= 12. \end{aligned}$$

Remember, add (-3) times the first equation to the second to create a new equation in only y? This leads to the solution $x = 4$, $y = 0$, or $\vec{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

Question 2. [20 points] Each of the two definite integrals below is improper. Choose ONLY ONE of them. For this choice, either show that the integral does not exist, or calculate its value:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{\cos x}}\, dx.$

Solution: This integral is improper since the integrand has a vertical asymptote at the upper limit. Hence we correct this with the limit:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{\cos x}}\, dx = \lim_{b\to\frac{\pi}{2}^-} \int_0^{b} \frac{\sin x}{\sqrt{\cos x}}\, dx.$$

The latter integrand is a regular definite integral and the integrand is continuous on the entire closed interval between the limits. To solve, a simple substitution is useful. Choose $u = \cos x$, so that $du = -sinx\, dx$. Neglecting the limits for a minute, the antiderivative of the integrand is

$$\int \frac{\sin x}{\sqrt{\cos x}}\, dx = \int \frac{-1}{\sqrt{u}}\, du = -2\sqrt{u} + C = -2\sqrt{\cos x} + C.$$

Hence

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{\cos x}}\, dx &= \lim_{b\to\frac{\pi}{2}^-} \int_0^{b} \frac{\sin x}{\sqrt{\cos x}}\, dx \\ &= \lim_{b\to\frac{\pi}{2}^-} \left(-2\sqrt{\cos x}\right)\Big|_0^b = \lim_{b\to\frac{\pi}{2}^-} \left(-2\sqrt{\cos b} + 2\sqrt{\cos 0}\right) = 0 + 2 = 2.\end{aligned}$$

(b) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\, dx.$

Solution: The problem here is not that the integrand is ill-defined at one of the limits. It is that the integrand has a vertical asymptote in the middle of the interval of integration; at $x = 0$. hence this integral is also improper for this reason. Really, properly written,

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\, dx = \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{\sqrt[3]{x}}\, dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{\sqrt[3]{x}}\, dx.$$

Inside the limits, the integrals are now fine. One can notice that the integrand is an odd function $f(-x) = -f(x)$, and thus is rotationally symmetric. This is not necessary, really, but it does mean that the two integrals on the right will be the same up to sign. This means that either they both exist or they don't. We will solve only one of them here: we get

$$\lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{\sqrt[3]{x}}\, dx = \lim_{b\to 0^-} \int_{-1}^{b} x^{-\frac{1}{3}}\, dx = \lim_{b\to 0^-} \left(x^{\frac{2}{3}}\right)\Big|_{-1}^{b} = \lim_{b\to 0^-} b^{\frac{2}{3}} - (-1)^{\frac{2}{3}} = 0 - 1 = -1.$$

Since this one exists, the other one will also and be of the other sign. Hence we will get

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\, dx = \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{\sqrt[3]{x}}\, dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{\sqrt[3]{x}}\, dx = (-1) + 1 = 0.$$

Question 3. [20 points] Solve the first-order, separable initial value problem defined for $t \geq 0$ given by

$$\frac{dy}{dt} = ty^2 - y^2 \sin t, \quad y(0) = \frac{1}{2}.$$

Solution: First, the right-hand-side can be rewritten $ty^2 - y^2 \sin t = (t - \sin t)\, y^2$. Thus the expression is "separable" into factors, each of which is not an expression involving the other variable. Then the differential equation can be written as

$$\frac{1}{y^2}\frac{dy}{dt} = (t - \sin t)\,, \text{ or in differential notation } \frac{1}{y^2}\, dy = (t - \sin t)\ dt.$$

This last one is the form most likely seen in books. Solving is simply integrating at this point, and in the latter form, one simply integrates with respect to each variable on each side:

$$\int \frac{1}{y^2}\ dy = \int (t - \sin t)\ dt$$
$$-\frac{1}{y} = \frac{t^2}{2} + \cos t + C.$$

Recall that the constant is necessary at this point since we have produced antiderivatives of each side, and whenever two function are equal, their respective antiderivatives are ONLY equal up to some constant! Really, as this stands, this IS the solution, written with y as an implicit function of t. This becomes the general solution to the differential equation. At this point, one needs only to find the value of C that corresponds to the initial data $y(0) = \frac{1}{2}$. We get

$$-\frac{1}{\frac{1}{2}} = -2 = \frac{0^2}{2} + \cos 0 + C = 1 + C,$$

and we find that $C = -3$. But it "feels" better to actually first solve for y explicitly as a function of t first and then solve for the constant C. Done this way, we get the general solution is

$$y(t) = \frac{-1}{\frac{t^2}{2} + \cos t + C}.$$

Then

$$y(0) = \frac{1}{2} = \frac{-1}{\frac{0^2}{2} + \cos 0 + C}$$

means $-2 = 1 + C$, or $C = -3$. The particular solution here is

$$\boxed{y(t) = \frac{-1}{\frac{t^2}{2} + \cos t - 3}}.$$

Question 4. [15 points] Let

$$\frac{dy}{dx} = y(y^2 - 1)(6 - y)$$

be a first-order autonomous differential equation. WITHOUT solving this differential equation, do the following:

(a) Find and classify the stability of all equilibrium solutions.

Solution: First, the warning NOT to try to solve the differential equation is one to heed. It is doable (by a partial fraction decomposition of the integrand) but not needed and quite tedious. Rather, since the differential equation is autonomous, look for all places where the derivative is 0. These are the equilibria. Here solve $y(y^2-1)(6-y) = 0$, which leads immediately to $y = 0$, $y = -1$, $y = 1$, and $y = 6$. Hence the equilibria solutions are $y(t) \equiv 0$, $y(t) \equiv \pm 1$, and $y(t) \equiv 6$. Note that the solution $y(t) \equiv -1$ is often missed.
To classify, simply look to see how all solution behave on each side of each equilibrium. This is done really by noticing that for each interval of the variable y whose endpoints are equilibria, the sign of the expression $f(y) = y(y^2 - 1)(6 - y)$ does not change. This means that for any starting value $y(0)$ of a solution, the derivative $\frac{dy}{dx}$ will not change sign forever. And since solutions cannot cross, every solution that starts in any interval bounded by equilibria must stay in that interval. For example, suppose we start a solution at $y(0) = \frac{1}{2}$. Then for all time, $y(t)$ will have to stay in the interval $(0, 1)$, since both $y = 0$ and $y = 1$ are equilibria. Furthermore, on this interval $f\left(\frac{1}{2}\right) < 0$. Hence here the solution $y(t)$ will tend toward the equilibrium solution $y(t) \equiv 0$ as t goes to infinity. This means that the equilibrium solution $y(t) \equiv 0$ is stable on this side. Playing this through on each side of each equilibrium, we find that $y(t) \equiv 0$ and $y(t) \equiv 6$ are stable, and $y(t) \equiv -1$ and $y(t) \equiv 1$ are unstable.

(b) Discuss the long term behavior of the solution to this differential equation that satisfies the initial value $y(0) = 3$. (That is, for the solution to this differential equation that satisfies $y(0) = 3$, what is $\lim_{x\to\infty} y(x)$.)

Solution: Based on the above discussion, this part becomes easy. Since the equilibria $y(t) \equiv 1$, is unstable, and $y(t) \equiv 6$ is stable, and since there are no other equilibria on the interval $(1, 6)$, ANY solution that starts in this interval will stay in this interval and tend toward the stable equilibrium $y(t) \equiv 6$. Since $y(0) = 3$, we have

$$\lim_{t\to\infty} y(t) = 6.$$

Question 5. [15 points] Do the following:

(a) In $\mathbb{R}^2$, find the equation of the line perpendicular to the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ that passes through the point $(4, 3)$.

Solution: The line perpendicular to a point (x_0, y_0) and perpendicular to a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is given by $a(x - x_0) + b(y - y_0) = 0$. Recall, that this is the dot product of two vectors, one of which is the one given and the other is a vector constructed to live on the line. The dot product is 0 to reflect the perpendicular nature of the vector directions. Here $x_0 = 4$, $y_0 = 3$, $a = 1$ and $b = 2$. The final answer is

$$\boxed{(x - 4) + 2(y - 3) = 0.}$$

(b) Parameterize this line via the parameter t. For your parameterization, what are the x and y values that correspond to the value $t = 2$.

Solution: Solving your answer above for y as a function of x, you would get $y = \frac{1}{2}x + 5$. One parameterization easily found is to simply let $x = t$ (this is like letting the x parameter be the line parameter). Then the parameterization is

$$\begin{aligned} x(t) &= t \\ y(t) &= -\frac{1}{2}t + 5. \end{aligned}$$

it is not insightful, but it is correct, and ANY parameterization works here. Given this parameterization, then for $t = 2$, we get $x - 2$, and $y = -\frac{1}{2}(2) + 5 = 4$.

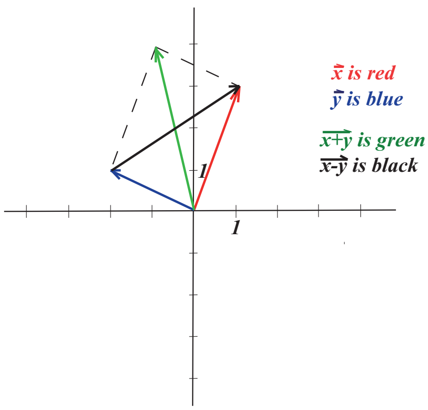
A different way is to use the vector given explicitly, and write a vector parameterization: Let $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Then the parameterization is simply a sum of scaled vectors

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

And for this parameterization, $t = 2$ on this line has coordinates $x(2) = 4 + (2)1 = 6$, and $y(2) = 3 + (2)2 = 7$.

Question 6. [15 points] Let $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\vec{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(a) Graph $\vec{x}$, $\vec{y}$, $\vec{x}+\vec{y}$ and $\vec{x}-\vec{y}$ on the graph provided.



(b) Calculate $|\vec{x}|$ and find the unit vector in the direction of $\vec{x}$.

Solution: The length of the vector is simply the distance from its base at the origin to its tip at the point $(1, 3)$. Hence

$$|\vec{x}| = \text{length}(\vec{x}) = \sqrt{1^2 + 3^2} = \sqrt{10}.$$

Any vector in the same direction of $\vec{x}$ is simply a multiple of $\vec{x}$, and if we divide the vector $\vec{x}$ by its length, we get the unit vector in the same direction. Hence

$$\text{unit vector in the direction of } \vec{x} = \frac{\vec{x}}{|\vec{x}|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}.$$

(c) Given $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, are the vectors $A\vec{x}$ and $\vec{y}$ linearly independent? Explain.

Solution: Here,

$$A\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

Now, two vectors are considered independent if and only if one is not a multiple of the other. Thus $\vec{y}$ is independent of $A\vec{x}$ as long as there is not a $c \in \mathbb{R}$, where $c\vec{y} = A\vec{x}$. But there is: let $c = -2$. Then

$$c\vec{y} = (-2) \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = A\vec{x}.$$

Hence $\vec{y}$ and $A\vec{x}$ are dependent.