FINAL EXAM: CALC II (BIO AND SOC. SCI.)

PROBLEM 1. The following three questions are unrelated.

a) Find the solution to the equation $\frac{dy}{dt} = 3t - 1$ which satisfies $y(2) = 5$.

b) Solve the equation $\frac{dy}{dt} = 2(1 - y)$, with $y(0) = 2$.

c) The autonomous equation $\frac{dy}{dt} = y^5$ has exactly one equilibrium solution. Determine wether it is a *stable* or *unstable* equilibrium.

PROBLEM 2.

a) Determine which of the following matrices is invertible and find its inverse.

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

b) With A as above, find X such that $AX = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

PROBLEM 3. In the xy-plane, consider the triangle that has vertices at coordinates $P(1, 2)$, $Q(4, 3)$ and $R(3, -4)$.

a) Determine the length of the segment QR.

b) Determine the angle at P (i.e. the angle RPQ).

c) Determine the equation of the line that passes through P and is perpendicular on the line QR.

PROBLEM 4.

a) Compute f_x and f_y for $f(x, y) = x \sin(\pi xy)$.

b) Consider the function $g(x, y) = x^2 y$. Determine the directional derivative of g at $(-1, 2)$ in the direction of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

c) With g as above, find a unit vector $\mathbf{v}$ such that the directional derivative of g at $(-1, 2)$ in the direction of $\mathbf{v}$ equals zero.

PROBLEM 5. Consider the function $f(x, y) = x^2 + y^2 - x + 2y$.

a) Determine and classify the critical points of f.

b) Find the absolute maxima and minima of f on the set

$$D = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$$

PROBLEM 6. Roll a fair dice twice.

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a) Find the probability that the minimum of the two numbers will be at least 4 (i.e. greater or equal to 4).

b) Find the probability that the first number is at least 5 given that the sum is at least 8.

c) Let S the sum of the two numbers. Find $E[S]$ and $\text{var}(S)$.

PROBLEM 7.

Assume that X and Y are two independent, normally distributed random variables: $X \sim N(40, 6)$ and $Y \sim N(50, 8)$.

a) Find the expected value and standard deviation of $X + Y$.

b) Find $P(X \geq 43)$ and $P(X \geq 34)$.

c) Determine the value of the integral $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2} dx$.

PROBLEM 8.

a) Toss a fair coin 6 times. Find the probability that you will get exactly 4 tails.

b) Assume that the height of the population ABC has mean 5 and standard deviation 1 (measured in ft). Take a random sample of 400 independent individuals. Take the average height of the people in the sample. This is the sample average. Determine the probability that the sample average is within 1% of the population mean.

Useful facts.

1. The notation $Y \sim N(\mu, \sigma)$ means that the random variable Y has normal distribution with mean μ and standard deviation σ. The probability density function of Y is $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, $-\infty < x < +\infty$.

2. If $Z \sim N(0, 1)$ has standard normal distribution, then

$$P(Z \leq 0.1) = .5398$$
$$P(Z \leq 0.2) = .5793$$
$$P(Z \leq 0.3) = .6179$$
$$P(Z \leq 0.4) = .6554$$
$$P(Z \leq 0.5) = .6915$$
$$P(Z \leq 1) = .8413$$
$$P(Z \leq 2) = .9772$$