pages	$2-3$	$4-6$	$7-10$	$11-13$	$14-16$	$17-20$	*total*
scores							

Final Exam, Dec 15, Calculus III, Fall, 2006, Eugene Ha and W. Stephen Wilson

I agree to complete this exam without unauthorized assistance from any person, materials or device.

Name: ______________________________ Date: ______________

TA Name and section: ______________________

NO CALCULATORS, NO PAPERS, SHOW WORK. (78 points total) To use cylindrical coordinates we have $\iiint f(x,y,z)dxdydz = \iiint f(r\cos\theta, r\sin\theta, z)rdrd\theta dz$.

To use spherical coordinates we have

$\iiint f(x,y,z)dxdydz = \iiint f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi$.

To evaluate the average value of a function on a curve, surface or solid, you integrate the function over the object and divide by the length/area/volume of the object.

1. (2 points) Find the directional derivative of the function $g(x, y, z) = x^2y + yz + x - z$, in the direction $\mathbf{j} = (0, 1, 0)$ at an arbitrary point (x, y, z).

2. (4 points total) Find the direction that the directional derivative of the function $g(x, y, z) = x^2y + yz + x - z$, is maximal at the point $(1, 1, 1)$ (2 points) and evaluate that derivative (2 points).

3. (6 points total) Let $z = f(x, y)$. Consider z as a function of the polar coordinates (r, θ) (i.e. $(x, y) = (r\cos(\theta), r\sin(\theta))$). Compute $\partial z/\partial r$, $\partial z/\partial \theta$, and $\partial^2 z/\partial r \partial \theta$ in terms of the partials of f with respect to x and y (2 points each).

4. (4 points total) Use the method of Lagrange multipliers to find the minimum and maximum values of the function $f(x, y) = x/2 + y/5$ subject to the constraint $x^2 + y^2 - 1 = 0$. (2 points for setting up the correct equations and 2 points for solving and interpreting them.)

5. (4 points total) Consider the hemisphere, $x^2 + y^2 + z^2 = 1$, $z \geq 0$. What is the average height (z-height, i.e. distance from the xy-plane) of points on this hemisphere when considered as a function on the disk $x^2 + y^2 \leq 1$ in the plane? You can use the fact that the area of the circle of radius 1 is π without having to compute it. (2 points for setting up the integral and 2 for evaluation)

6. (4 points total) Consider the hemisphere, $x^2 + y^2 + z^2 = 1$, $z \geq 0$. What is the average height (z-height, i.e. distance from the xy-plane) of points on this hemisphere when considered as a function on the hemisphere itself. You can use the fact that the area of the hemisphere is 2π without having to compute it. (2 points for setting up the integral and 2 for evaluation)

7. (2 points) Consider the mapping, $T(u, v) = (x, y)$ defined by the equations $x = u + v$, $y = v - u^2$. Compute the derivative of this function.

8. (2 points) Compute the Jacobian determinant $\partial(x, y)/\partial(u, v)$ of the function T in the previous problem.

9. (2 points) A triangle region W in the uv-plane has vertices $(0, 0)$, $(0, 1)$, $(1, 0)$. Describe, by means of a sketch, its image, S, in the xy-plane, under the mapping T from the previous page. (Hint: What is the image of the boundary of W?)

10. (4 points total) Express the area of S (from the previous page) as an integral over S (2 points) and compute the integral (2 points).

11. (4 points total) Express the area of S (from the previous pages) as an integral over W (2 points) and compute the integral (2 points).

12. (10 points total) Let $f(x,y) = (3-x)(3-y)(x+y-3)$. (a) Find all (4) of the critical points of f (1 point each), and for each one, determine whether it's a (relative) minimum, a (relative) maximum, or a saddle point and give your reasons (1 point each). (It may be helpful to sketch the solution to the equation $f(x,y) = 0$ (and others).) (b) Does f achieve an absolute minimum or absolute maximum value? Justify your answer (2 points).

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13. (4 points total) Consider the solid hemisphere, $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$ of uniform density. By symmetry the center of mass is on the z-axis. What is its z-coordinate? (Another way to look at this is that this is the average height of all the points in half a solid ball viewed as a function defined on the solid hemisphere.) You can use the fact that the volume of the solid unit hemisphere is $\frac{2\pi}{3}$ without having to compute it. (2 points for setting up the integral and 2 for evaluation)

14. (10 points total) Let C be the curve in the xy-plane which is the union of the curve $4x^2+y^2-4=0$, $y \geq 0$ and the line segment $|x| \leq 1$, $y=0$. Orient C in the usual counter-clockwise direction. Let $\vec{F}$ be the vector field $(2+3yx)\vec{i}+16y\vec{j}$. (a) Compute the (oriented) line integral $\int_C \vec{F} \cdot d\vec{s}$ directly by using a parameterization of C (One option is to think polar coordinates. If $x = r\cos(t)$, what would y be?)(2 points for parameterizing the curve. 2 points for setting up the integral. 2 points for doing the integral). (b) This integral is, by Green's theorem, equal to some double integral. Set that double integral up (2 points). Compute the integral (2 points).

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15. (2 points) Let C be the intersection of the surface $x^2 + y^2 + z^2 = 1$ with the surface $x + y + z = 0$. Evaluate the line integrals $\int_C (x^2 + y^2 + z^2)\, ds$, $\int_C (x^2)\, ds$, $\int_C (y^2)\, ds$, and $\int_C (z^2)\, ds$. Explain how you got your answers.

16. (8 points total) Let S be the surface of the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, and let $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$. Orient S so that its unit normal points outward. The surface integral $\iint_S \vec{F} \cdot d\vec{S}$ can be computed directly or, by using Gauss' theorem, there is a triple integral way to compute it. Set up the integrals both ways (2 points each) and evaluate both of them (2 points each).

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17. (6 points total) Let Q be the square $[0, \pi] \times [0, \pi]$. Compute the integral $\iint_Q |\cos(x+y)|\,dx\,dy$. (2 points for describing where $\cos(x+y)$ is positive and negative, 2 points for setting up the integral, and 2 points for evaluating.)

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