Final Exam, Linear Algebra, Fall, 2003, W. Stephen Wilson

Name: ______________________________

TA Name and section: ______________________________

NO CALCULATORS, SHOW ALL WORK, NO OTHER PAPERS ON DESK.

There is very little actual work to be done on this exam if you know what you are doing and can use the work which has already been done. Consequently it is very very important not to mess up the caculations you really do have to do. Check them, otherwise all of the rest of the work will be wrong.

We will be working with the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ for quite awhile. (Problems 1-20.)

(1) (3 points) The matrix A is symmetric. What is the definition of a *symmetric* matrix?

(2) (3 points) A symmetric matrix is *orthogonally diagonalizable.* What is the definition of *orthogonally diagonalizable*?

(3) (3 points) What is the *rank* of A?

(4) (3 points) Come to think of it, what is the definition of *rank*?

(5) (3 points) Calculate A^2.

(6) (3 points) What is the trace of A?

(7) (3 points) What is the determinant of A?

(8) (3 points) Solve the equation $Ax = 0$.

(9) (3 points) If possible, solve the equation $Ax = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. If not, then find all *least squares solutions* for this equation.

(10) (3 points) Find a basis for the kernel of A.

(11) (3 points) Find a basis for the image of A.

(12) (3 points) Find the characteristic polynomial for A.

(13) (3 points) Find the Eigenvalues for A.

(14) (3 points) For A, find an Eigenvector for each of the Eigenvalues. To make it easier to grade, choose Eigenvectors with integer coordinates where the integers are as small as possible.

(15) (3 points) Use your Eigenvectors to make a basis of $\mathbb{R}^3$. Chose the first basis vector to be the Eigenvector associated with the largest Eigenvalue and the third basis vector to be the Eigenvector associated with the smallest Eigenvalue. Call this basis $\mathcal{B}$. We have a linear transformation given to us by A. What is the matrix D when we use coordinates from this new basis, i.e. $D : [x]_{\mathcal{B}} \to [Ax]_{\mathcal{B}}$?

(16) (3 points) We know there is a matrix S such that $S^{-1}AS = D$ (for the basis $\mathcal{B}$). Find S.

(17) (3 points) Find S^{-1}.

(18) (3 points) Modify the basis $\mathcal{B}$ so you can orthogonally diagonalize A.

(19) (3 points) Find the new S needed to orthogonally diagonalize A (same D).

(20) (3 points) Find the inverse of this last S.

We will now study the matrix $C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. The book tells us that there is an orthonormal basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$ and an orthonormal basis $\{u_1, u_2\}$ for $\mathbb{R}^2$ such that $Cv_1 = \sigma_1 u_1$, $Cv_2 = \sigma_2 u_2$, and $Cv_3 = 0$ (with $\sigma_1 \geq \sigma_2$). We will study this situation for a bit. (Problems 21-24.)

(21) (3 points) Find σ_1 and σ_2.

(22) (3 points) Find v_1, v_2 and v_3.

(23) (3 points) Find u_1 and u_2.

(24) (3 points) Give the *Singular-value decomposition* (SVD) of C.

Consider the quadratic form $q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 2x_2^2 + 2x_2x_3 + x_3^2$ for the next few problems. (Problems 25-31.)

(25) (3 points) Find a symmetric matrix A such that $q(x) = x^T Ax$.

We have a theorem that says there is an orthonormal basis u_1, u_2, u_3, such that using this coordinate system, $q(c) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2$.

(26) (3 points) Find λ_1, λ_2, and λ_3. Put them in decreasing order.

(27) (3 points) Find u_1, u_2, and u_3.

(28) (3 points) Setting $q(x) = 1$ we have a surface in $\mathbb{R}^3$. Find the principal axes.

(29) (3 points) Find, in our new coordinates, c, the closest point on the surface $q(c) = 1$ to the origin.

(30) (3 point) Find, in our old coordinates, x, the closest point on the surface $q(x) = 1$ to the origin.

(31) (3 points) This surface has a fairly simple description. What does it look like?

(32) (3 points) Let A be an upper triangular $n \times n$ matrix with determinant equal to 3. Multiply by 5 all terms in the matrix above and to the right of the diagonal (but not on the diagonal). What is the determinant of the new matrix?

(33) (3 points) State Cramer's rule.

(34) (3 points) If A is invertible, what is A^{-1} in terms of the adjoint (which you should define of course)?

We will be working with P_3, the set of all polynomials with degree less than or equal to 3. We will think of them as a subspace of $C[-1,1]$, the continuous functions from the interval $[-1,1]$ to the reals. P_3 has an inner product given by $< f, g >= \int_{-1}^{1} f(x)g(x)dx$. P_3 has what we will call a *standard* basis: $\{1, x, x^2, x^3\}$. Let V be the subspace spanned by $\{1, x\}$. (Problems 35-41.)

(35) (3 points) Find a basis for the orthogonal complement of V, $V^{\perp}$.

(36) (3 points) Find an orthonormal basis for V.

(37) (3 points) What is the matrix of the orthogonal projection of $P_3 \to V \subset P_3$ with respect to the standard basis of P_3. (It should be a 4×4 matrix.)

(38) (3 points) If we have an orthonormal basis of P_3, $\{u_1, u_2, u_3, u_4\}$, where u_1 and u_2 form a basis for V and u_3 and u_4 form a basis for $V^{\perp}$, then find the matrix for the orthogonal projection to V with respect to this basis.

(39) (3 points) Evaluate the orthogonal projection to V of the polynomial $1 + 2x + 3x^2 + 4x^3$.

(40) (3 points) What is the "least squares" linear approximation to x^3 on the interval $[-1, 1]$?

(41) (3 points) In problem 40, what is the least squares integral that is minimized?

Let $\{u_1, u_2, u_3\}$ be an orthonormal basis, $\mathcal{B}$, for $\mathbb{R}^3$. Define a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$, by $L\begin{pmatrix} a \\ b \\ c \end{pmatrix} = au_1 + bu_2 + cu_3$. (Problems 42-46.)

(42) (3 points) What is the matrix A for the linear transformation L with respect to the standard basis?

(43) (3 points) What is the matrix B for the linear transformation L with respect to the basis $\mathcal{B}$?

(44) (3 points) What is the change of basis matrix S so that $B = S^{-1}AS$?

(45) (3 points) Let V be the subspace spanned by u_1 and u_3. What is the matrix for the orthogonal projection to V with respect to our basis $\mathcal{B}$?

(46) (3 points) Let V be the subspace spanned by u_1 and u_3. What is the matrix for the orthogonal projection to V with respect to the basis $\{u_1, u_1 + u_2, u_2 + u_3\}$?