# The second real Johnson-Wilson theory and non-immersions of $RP^n$

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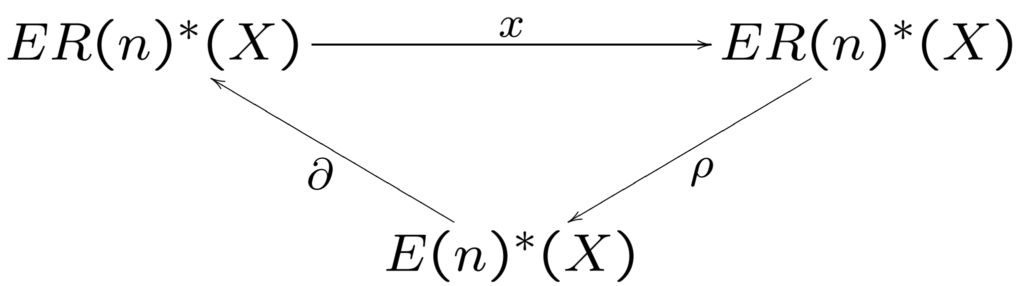
$E(n)$ is $|v_n| = 2(2^n - 1)$ periodic.

$ER(n)$ is $|v_n^{n+1}| = 2^{n+2}(2^n - 1)$ periodic.




The degree of $x$ is $-\lambda(n) = -2^{2n+1} + 2^{n+2} - 1$

$$x^{2^{n+1}-1} = 0$$

For $n = 1$ $|x| = -2^3 + 2^3 - 1 = -1$ and $x = \eta$ because $ER(1) = KO_{(2)}$ and $E(1) = KU_{(2)}$.

For $n = 1$ periodicity is $2^3(2^1 - 1) = 8$.

$KU_{(2)} = E(1)$ is 2-periodic. Grade all over $\mathbb{Z}/(8)$.

Compute $KO^*_{(2)}$ from $KU^*_{(2)}$.

We know the answer: (Graded over $\mathbb{Z}/(8)$.)

Free $\mathbb{Z}_{(2)}$ on 1 in degree 0.

Free $\mathbb{Z}_{(2)}$ on $\beta$ in degree $-4$.

$\mathbb{Z}/(2)$ on $\eta$ in degree $-1$.

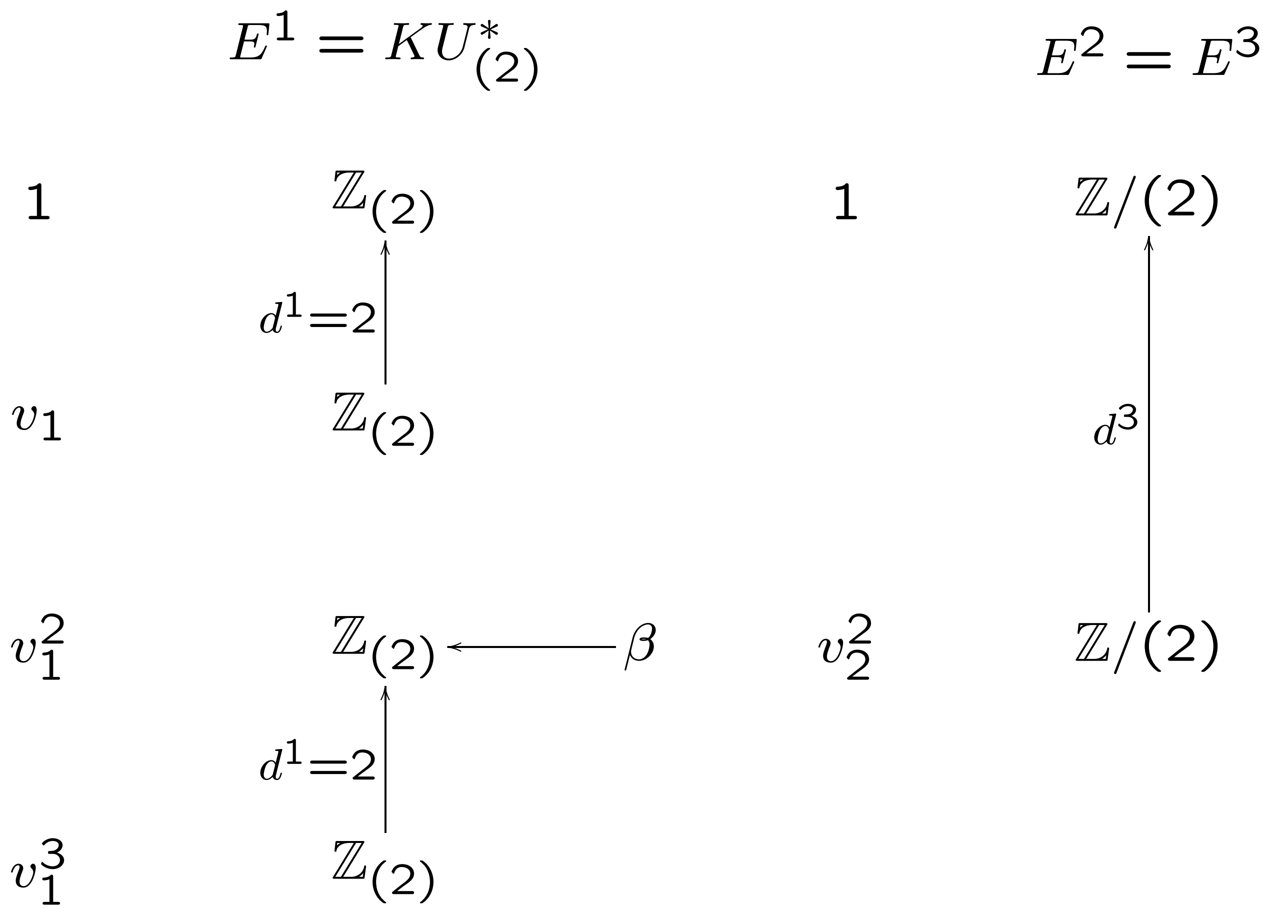
$\eta^3 1 = 0$ $\eta\beta = 0$ and $\beta^2 = 4$.

Only 3 differentials because $x = \eta$ has $x^3 = 0$.

Degree of $d^r$ is $r + 1$.

$KU^*_{(2)}$ is $\mathbb{Z}_{(2)}$ free on $v_1^i$, $0 \leq i < 4$.

Set $v_1^4 = 1$. $|v_1| = -2$.



$E^1 = KU^*_{(2)}$
$E^2 = E^3$
1
$\mathbb{Z}_{(2)}$
$d^1{=}2$
$v_1$
$\mathbb{Z}_{(2)}$
$v_1^2$
$\mathbb{Z}_{(2)}$
$\beta$
$d^1{=}2$
$v_1^3$
$\mathbb{Z}_{(2)}$
1
$\mathbb{Z}/(2)$
$d^3$
$v_2^2$
$\mathbb{Z}/(2)$

Facts about $ER(2)$. $|x| = -17$. $x^7 = 0$.

$E(2)^* = \mathbb{Z}_{(2)}[v_1, v_2^{\pm 1}]$.

$ER(2)$ is 48 periodic, so for $E(2)$ we set $v_2^8 = 1$.

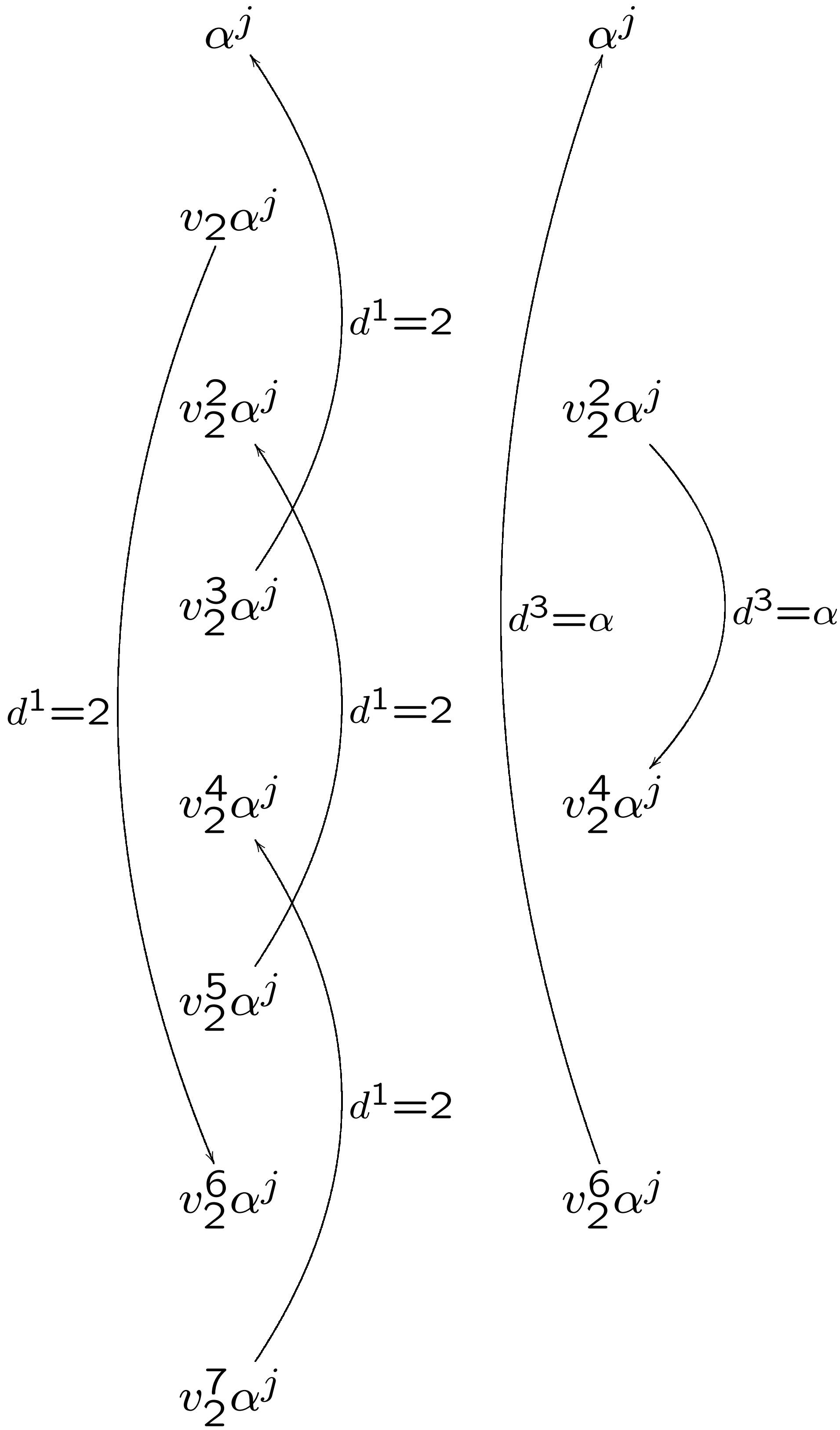
(Recall $|v_2| = -6$.) Index over $\mathbb{Z}/(48)$.

No $v_1$ in $ER(2)^*$ but there is an $\alpha \in ER(2)^{-32}$.

$\alpha \longrightarrow v_2^5 v_1$. Replace $v_1$ with $\alpha \in E(2)^*$.

$E(2)^*$ is $\mathbb{Z}_{(2)}$ free on $v_2^i \alpha^j$, $0 \le i < 8$, $0 \le j$.

Compute $ER(2)^*$ from $E(2)^*$.


$\alpha^j$
$v_2\alpha^j$
$d^1=2$
$v_2^2\alpha^j$
$v_2^3\alpha^j$
$d^1=2$
$d^1=2$
$v_2^4\alpha^j$
$v_2^5\alpha^j$
$d^1=2$
$v_2^6\alpha^j$
$v_2^7\alpha^j$
$\alpha^j$
$v_2^2\alpha^j$
$d^3=\alpha$
$d^3=\alpha$
$v_2^4\alpha^j$
$v_2^6\alpha^j$

As differentials get harder, there is less to deal with!

We want applications.

James says: If $RP^{2n}$ immerses ($\subseteq$) in $\mathbb{R}^{2k}$ then there exists an axial map:

$$RP^{2n} \times RP^{2^K-2k-2} \longrightarrow RP^{2^K-2n-2}.$$

Don Davis uses $BP$, or, really, $BP\langle 2\rangle^*(-)$.

$BP\langle 2\rangle^* \simeq \mathbb{Z}_{(2)}[v_1, v_2]$.

There is no $v_2$ torsion so we can invert $v_2$ and use $E(2)^*(-)$.

$$E(2)^*(RP^{2^K-2n-2})$$

$$\downarrow$$

$$E(2)^*(RP^{2n})$$

$$\bigotimes_{E(2)^*}$$

$$E(2)^*(RP^{2^K-2k-2})$$

$x_2^{2^{K-1}-n} = 0$ maps non-trivially for

$n = m + \alpha(m) - 1$ and

$k = 2m - \alpha(m)$.

Don shows $RP^{2n} \not\subseteq \mathbb{R}^{2k}$ for these $n$ and $k$.

To do same with $ER(2)^*(-)$ we will need

$ER(2)^*(RP^{2n})$.

$E^1$ of spectral sequence is $E(2)^*(RP^{2n})$

$v_2^s \alpha^k u^j$ in a 2-adic basis.

$0 \leq s < 8$

$0 \leq k$

$0 < j \leq n$

$u$ is not Don's $x_2$.

There is a $u \in ER(2)^{-16}(RP^{2n})$

which maps to $v_2^3 x_2$.

We use this $u$.

$d^1$ is easy.

$d^3$ follows from

$RP^{2n-2} \to RP^{2n} \to RP^{2n}/RP^{2n-2}$.

Only have $d^{\{1,3,5,7\}}$ because even degree.

After $d^3$ have $u^{\{1,2,3\}}$ and $v_2^4 u^{\{1,2,3\}}$

There is a known

$d^7 : v_2^4 u^{\{1,2,3\}} \to u^{\{1,2,3\}}$

Differentials are hard now, but not much left.

For $n = 0, 3, 4, 7 \bmod 8$.

$$v_2^2 u^{n-1} \xrightarrow{u} v_2^2 u^n$$

$$v_2^3 u^n$$

$$v_2^6 u^{n-1} \xrightarrow{u} v_2^6 u^n$$

$$v_2^7 u^n$$

For $n = 1, 2, 5, 6 \bmod 8$.

$$\begin{array}{ccc} & & v_2^1 u^n \\ & \overset{d^5}{\nearrow} & \\ v_2^2 u^{n-1} & \xrightarrow{u} & v_2^2 u^n \end{array}$$

$$\begin{array}{ccc} & & v_2^5 u^n \\ & \overset{d^5}{\nearrow} & \\ v_2^6 u^{n-1} & \xrightarrow{u} & v_2^6 u^n \end{array}$$

Element of interest:

$$x^2 \alpha^k v_2^5 u^n = \alpha^k u^{n+1} \neq 0.$$

$KU^0(RP^{2n})$ has $u^n \neq 0$ and $u^{n+1} = 0$.

For $n = 0, 3$ mod 4.

$KO^0(RP^{2n})$ has $u^n \neq 0$ and $u^{n+1} = 0$.

For $n = 1, 2$ mod 4.

$KO^0(RP^{2n})$ has $u^{n+1} \neq 0$ and $u^{n+2} = 0$.

$E(2)^{16*}(RP^{2n})$ has $\alpha^k u^j$ $0 < j \leq n$.

**Theorem 1.** *$ER(2)^{16*}(RP^{2n})$ consists of the elements $\alpha^k u^j$, with $0 \leq k$ and $0 < j \leq n$, and, when*

*$n = 0$ or 7 modulo 8, no others,*

*$n = 1$ or 6 modulo 8, $\alpha^k u^{n+1}$,*

*$n = 2$ or 5 modulo 8, $\alpha^k u^{n+1}$, and $u^{n+2}$,*

*$n = 3$ or 4 modulo 8, $u^{n+1}$, $u^{n+2}$, and $u^{n+3}$,*

We have, when $n = 0, 7 \bmod 8$:

$$ER(2)^{16*}(RP^{2n}) \simeq E(2)^{16*}(RP^{2n})$$

Purely algebraically, we have surjections

$$ER(2)^{16*}(RP^{2n}) \longrightarrow E(2)^{16*}(RP^{2n+2})$$

when $n = 1, 2, 5, 6 \bmod 8$.

Back to the axial maps.

$$
\begin{array}{ccc}
ER(2)^*(RP^{2^K-2n-2}) & \longrightarrow & E(2)^*(RP^{2^K-2n-2}) \\
\downarrow & & \downarrow \\
ER(2)^*(RP^{2n}) & & E(2)^*(RP^{2n}) \\
\bigotimes_{ER(2)^*} & \longrightarrow & \bigotimes_{E(2)^*} \\
ER(2)^*(RP^{2^K-2k-4}) & & E(2)^*(RP^{2^K-2k-2})
\end{array}
$$

When $n = 0, 7$ mod 8, top two are isomorphisms.

When $-k-2 = 1, 2, 5, 6$ mod 8, bottom is surjection.

Now we mooch off of Don to show it is non-zero in the tensor product.

**Theorem 2.** *When the pair* $(m, \alpha(m))$ *is, modulo 8,* (2, 7), (7, 2), (6, 3), (3, 6), (7, 1), (4, 4), (3, 5), *or* (0, 0), *then*

$$RP^{2(m+\alpha(m)-1)} \text{ does not immerse } (\not\subseteq)$$

$$\text{in } \mathbb{R}^{2(2m-\alpha(m)+1)}.$$

*When the pair* $(m, \alpha(m))$ *is, modulo 8,* (4, 3), (1, 6), (0, 7), *or* (5, 2), *then*

$$RP^{2(m+\alpha(m))} \not\subseteq \text{ in } \mathbb{R}^{2(2m-\alpha(m)+1)}.$$

An improvement of 1 or 2 for half of the $k$'s and 1/4 of the $n$'s, so for 1/8 of the cases he deals with.

$(m, \alpha(m)) = (6, 3)$ mod 8.

$$RP^{16+2^{i+1}} \not\subseteq \mathbb{R}^{20+2^{i+2}}$$

$RP^{48} \not\subseteq \mathbb{R}^{84}$ $\quad$ $RP^{80} \not\subseteq \mathbb{R}^{148}$ $\quad$ $RP^{144} \not\subseteq \mathbb{R}^{276}$.

$(m, \alpha(m)) = (4, 4)$ mod 8.

$$RP^{62+2^{i}} \not\subseteq \mathbb{R}^{106+2^{i+1}}$$

$RP^{126} \not\subseteq \mathbb{R}^{234}$ $\quad$ $RP^{190} \not\subseteq \mathbb{R}^{362}$.

The pair $(4, 3)$ mod 8 gives

$$RP^{14+2^{i+1}+2^{j+1}} \not\subseteq \mathbb{R}^{12+2^{i+2}+2^{j+2}}.$$

$RP^{62} \not\subseteq \mathbb{R}^{108}$ $\quad$ $RP^{94} \not\subseteq \mathbb{R}^{172}$ $\quad$ $RP^{158} \not\subseteq \mathbb{R}^{300}$.

$RP^{110} \not\subseteq \mathbb{R}^{204}$ $\quad$ $RP^{174} \not\subseteq \mathbb{R}^{332}$.

Unfortunately, the tensor product is not enough.

$$E(2)^*(RP^{2^K-2n-2})$$
$$\downarrow$$
$$E(2)^*(RP^{2n})$$
$$\bigotimes_{E(2)^*}$$
$$E(2)^*(RP^{2^K-2k-2})$$
$$\downarrow$$
$$E(2)^*(RP^{2n} \times RP^{2^K-2k-2})$$

For $E(2)^*(-)$ this last map is an injection from C-F 1964.

Nothing like that for $ER(2)^*(-)$.

Two kinds of problems.

First:

Perhaps image of $u^{2^{K-1-n}}$ is $Z+xW$, with $Z$ going to non-zero in $E(2)^*(-)$ but $Z+xW$ going to zero in $ER(2)^*$(product).

$$
\begin{array}{ccc}
ER(2)^*(RP^\infty) & \longrightarrow & E(2)^*(RP^\infty) \\
\downarrow & & \downarrow \\
ER(2)^*(RP^\infty) & & E(2)^*(RP^\infty) \\
\bigotimes_{ER(2)^*} & \longrightarrow & \bigotimes_{E(2)^*} \\
ER(2)^*(RP^\infty) & & E(2)^*(RP^\infty)
\end{array}
$$

These are all isomorphisms in degrees $16*$.

We have Kunneth theorems for $RP^\infty$ for both $ER(2)^*(-)$ and $E(2)^*(-)$.

There is no $xW$. The coproduct is exactly the same for both theories.

This is very special to $16*$.

$2u +_F \alpha u^2 +_F u^4 = 0$

Next we need to show that our obstruction is non-zero when we map

$$ER(2)^*(RP^{2^K-2n-2})$$

$$\downarrow$$

$$ER(2)^*(RP^{2n})$$

$$\bigotimes_{ER(2)^*}$$

$$ER(2)^*(RP^{2^K-2k-4})$$

$$\downarrow$$

$$ER(2)^*(RP^{2n} \times RP^{2^K-2k-4})$$

But we have no map

$$ER(2)^*(RP^{2n} \times RP^{2^K-2k-4}) \rightarrow$$

$$E(2)^*(RP^{2n} \times RP^{2^K-2k-2})$$

**Theorem 3.** *Let $m \leq n$, then*

$$BP^*(RP^{2m} \wedge RP^{2n}) \simeq$$

$$BP^*(RP^{2m}) \otimes_{BP^*} BP^*(RP^{2n})$$

$$\oplus \Sigma^{2n-1} BP^*(RP^{2m})$$

**Theorem 4.** *Let $m \leq n$, then*

$$E(2)^*(RP^{2m} \wedge RP^{2n}) \simeq$$

$$E(2)^*(RP^{2m}) \otimes_{E(2)^*} E(2)^*(RP^{2n})$$

$$\oplus \Sigma^{-16n-1} E(2)^*(RP^{2m})$$

*represented by (2-adic basis)*

$$v_2^s \alpha^k u_1^i u_2 \quad 0 \leq k \quad 0 < i \leq m \quad 0 \leq s < 8$$

$$v_2^s u_1^i u_2^j \quad 0 < i \leq m \quad 1 < j \leq n \quad 0 \leq s < 8$$

*and*

$$v_2^s \alpha^k u_1^j z_{-16n-17} \quad 0 \leq k \quad 0 \leq j < m \quad 0 \leq s < 8.$$

Because of the map $ER(2)^*(-) \to E(2)^*(-)$ we always have

$$\alpha^k u_1^i u_2$$

$$u_1^i u_2^j$$

for $i \leq m$ and $1 < j \leq n$.

For $n = 1, 2, 5, 6$ we also need $u_1^i u_2^{n+1}$.

By products, this would be

$$x^2 v_2^5 u_1^i u_2^n = u_1^i u_2^{n+1}$$

All we have to do is show that $v_2^5 u_1^i u_2^n$ is not in the image of $d^1$ or $d^2$.

$d^1$ is easy. $d^2$ is odd degree and we now have odd degree elements.

We show that $z_{-16n-17}$ is a real element and this prevents the $d^2$ hitting $v_2^5 u_1^i u_2^n$.