CALCULUS III: PROJECT 1B

1. Spherical Geometry

In this project, we will investigate non-Euclidean geometry, and in particular, the spherical geometry. You might have noticed that airplane flight paths do not look like straight lines on the map. That is because a shortest path between two points on a sphere consists of an arc of a great circle, i.e., the intersection of the sphere with a plane passing through the center of the sphere. Arcs of most great circles correspond to curved lines on a map. We will determine precisely what curves great circles correspond to for a particular map given by the stereographic projection.[1]

As a starting point, we will take the definition of the length of a path $c : [0,1] \to \mathbb{R}^2$

$$l(c) = \int_0^1 \|c'(t)\| dt.$$

Proposition 1. *In the Euclidean space $\mathbb{R}^2$, a straight line is the short path between two points.*

Proof. Suppose $c(t) : [0,1] \to \mathbb{R}^2$ is the shortest path between its endpoints. Since the length of a curve is invariant under translations and rotations, we may assume for simplicity that $c(0) = (0,0)$ and $c(1) = (A, 0)$ for some $A \geq 0$. Now, consider the path

$$d(t) = (c_1(t), 0)$$

where $c(t) = (c_1(t), c_2(t))$. We have

$$\begin{aligned} l(c) = \int_0^1 \|c'(t)\| dt &= \int_0^1 \sqrt{c_1'(t)^2 + c_2'(t)^2} dt \\ &\geq \int_0^1 \sqrt{c_1'(t)^2} dt \\ &= l(d) \end{aligned}$$

with equality holding if and only if $c_2(t) = 0$ for all t, i.e., if $c = d$. Since we assumed c was the shortest path between its endpoints and d has the same endpoints as c, it follows that the equality holds and $c = d$. In particular, c lies on the x axis and hence is a straight line. □

You will use a similar argument later to determine the shortest paths on a sphere.

Let $S^2 \subset \mathbb{R}^3$ be the unit 2-dimensional sphere centered at the origin. Define the stereographic projection map $\Phi : \mathbb{R}^2 \to S^2$ by the following construction. Given a point $(x, y) \in \mathbb{R}^2$, the line passing through $(x, y, 0)$ and the north pole, $(0, 0, 1)$, intersects the sphere in exactly one other point. We define $\Phi(x, y)$ to be this point.

Task 2. Find an explicit formula for Φ. What is the image of Φ?

Task 3. For $\boldsymbol{u} \in \mathbb{R}^2$ a two-dimensional vector, find $\|\boldsymbol{D}\Phi(x, y)\boldsymbol{u}\|$ in terms of $x, y, \|\boldsymbol{u}\|$. Interpret $\boldsymbol{D}\Phi(x, y)\boldsymbol{u}$ geometrically. [2]

[1] This is not the map which is usually used for earth.

[2] At some point in your computation, you might need to find $\|u_1\boldsymbol{a} + u_u\boldsymbol{b}\|^2 = u_1^2\|\boldsymbol{a}\|^2 + 2u_1u_2\boldsymbol{a}\cdot\boldsymbol{b} + u_2^2\|\boldsymbol{b}\|^2$ for appropriate vectors $\boldsymbol{a}, \boldsymbol{b}$. Your computation will become more manageable if you compute $\|\boldsymbol{a}\|, \|\boldsymbol{b}\|$, and $\boldsymbol{a}\cdot\boldsymbol{b}$ separately first.

Task 4. Let $c : [0, 1] \to \mathbb{R}^2$ be a curve in $\mathbb{R}^2$. Find the formula for the length of the corresponding curve $\Phi \circ c$ on the unit sphere. We will denote it by $l_{\text{Sphere}}(c)$. Task 3 will be useful.

Recall that the square of the arc length element ds^2 in the Euclidean plane is given by

$$ds^2_{\text{Euclid}} = dx^2 + dy^2.$$

Without getting bogged down in precise definitions, the above formula indicates that the length of a path $c(t)$ is obtained by integrating the one-form ds along the path, i.e.

$$l(c) = \int_0^1 \frac{ds}{dt} dt = \int_0^1 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt.$$

Task 5. Find the arc length element corresponding to the spherical length. It should have the form

$$ds^2_{\text{Sphere}} = f(x, y)dx^2 + g(x, y)dxdy + h(x, y)dy^2$$

for some functions f, g, h.

The space $\mathbb{R}^2$ together with ds_{Sphere} define everything we need to study geometry on a sphere[3] without ever having to consider the three-dimensional space we originally defined our sphere in. For example, the hyperbolic two-dimensional space, with which you might be familiar from Escher's art, can be described by the open unit disk $D \subset \mathbb{R}^2$ with the arc length element given by

$$ds^2_{\text{hyperbolic}} = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$

Unlike the sphere, the hyperbolic space cannot be described as a surface in the Euclidean three-dimensional space.

We are now in the position to show that the curve between two points on the sphere minimizing the length is an arc of a great circle. [4] Given a length minimizing curve, to show that it is an arc of a great circle, we will again use the symmetry of the sphere to assume that its starting point is at the south pole and the endpoint is on the plane $y = 0$.

Task 6. Let $c : [0, 1] \to \mathbb{R}^2$ be a path with $c(0) = (0, 0)$ and $c(1) = (A, 0)$ for some $A > 0$ minimizing the spherical length $l_{\text{Sphere}}(c)$ among all paths from $(0, 0)$ to $(A, 0)$. Show that c lies on the x-axis.[5]

Now that you've showed that the great circles are the length minimizing curves, your next task is to determine what they look like on the map. In other words, your task is to compute the image of the great circles under the inverse map Φ^{-1}.

Task 7. Describe the image of the great circles under the map Φ^{-1}. You might want to follow the following outline:

(1) A great circle is given by the intersection of the plane going through the origin having normal vector $\hat{n} = \langle a, b, c\rangle$.
(2) Consider first the case, $c = 0$ and describe the resulting curves.
(3) In the case $c \neq 0$, you may divide $\hat{n}$ by c and therefore assume $c = 1$. Describe the shape of the resulting curves.

[3]We will talk about shortest paths, but you can also define angles and area in terms of ds_{Sphere}.

[4]The general term for a length minimizing curve is a *geodesic*.

[5]Hints: Consider the path $d(t) = (\sqrt{c_1(t)^2 + c_2(t)^2}, 0)$ as we did in Proposition 1. You might need to use Cauchy-Schwarz inequality.

(4) Consider the intersection of the curves in the previous part with the unit circle $x^2 + y^2 = 1$.

For example, in the case of hyperbolic space which can be modelled by the open unit disk $D = \{(x, y)|x^2 + y^2 < 1\}$ with the arc length element given by $ds^2_{\text{hyperbolic}} = \frac{dx^2+dy^2}{(1-(x^2+y^2))^2}$, the geodesics are arcs of circles which meet the boundary at a right angle and straight line segments passing through the origin. The length of the geodesics is in fact infinite which comes from the fact that the arc length element at a point x tends to infinity as x approaches the boundary. You can verify it for yourself by computing the hyperbolic length of the vertical geodesic.

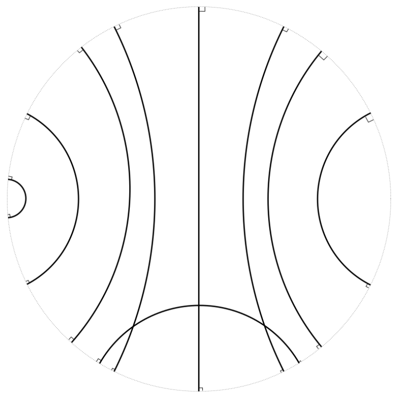

FIGURE 1.1. The Poincaré model of hyperbolic plane is the open disk with $ds^2_{\text{hyperbolic}} = \frac{dx^2+dy^2}{(1-(x^2+y^2))^2}$. The geodesic curves are arcs of circles meeting the boundary at a right angle.