CALCULUS III: GROUP PROJECT 2
ARCHIMEDES' PRINCIPLE

1. Overview

In this project we will study buoyancy force an object experiences when submerged in liquid. You have probably seen the statement of the Archimedes' principle in a physics class which relates the buoyancy force to the volume of the object. We will prove Archimedes' principle in the last section of the project, but first we will compute buoyancy with a double integral from the fact that air pressure decreases with altitude. We will use this computation in section 3 to "design" a hot-air balloon.

You may form new groups for this project or keep the same group you worked with for project 1. Your group should consist of 3-4 students from your section. Inform your TA of the groups you have formed no later than **April 16** and inform your TA if you would like assistance finding a group to work with. Written submission for the project is due on **May 2** on Gradescope. You will need to present your work to your TA during the last week of classes. Your TA will schedule the times for your to present. You need to be at least half-way done with the project by the time you present.

2. Buoyancy Force

If a polyhedron is submerged in a liquid with constant pressure P, then each of its faces R_i experiences force $\boldsymbol{F}_i$ in the direction normal to R_i with magnitude $PA(R_i)$ where $A(R_i)$ is the area of R_i.

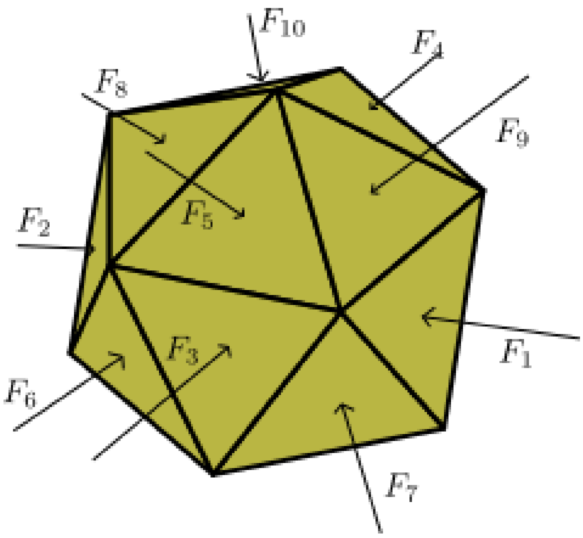


Figure 2.1. Force exerted on an icosahedron.

To find the total force exerted on the polyhedron, you would compute the sum

$$\boldsymbol{F} = \sum_i \boldsymbol{F}_i = \sum_i PA(R_i)\hat{\boldsymbol{n}}_i$$

where $\hat{\boldsymbol{n}}_i$ is the unit vector normal to face R_i pointing into the polyhedron.[1] Hopefully this expression reminds you of a surface integral.

In the presence of gravitational field, the pressure of liquid is not constant: it decreases with altitude. This is the cause of buoyancy force an object experiences when submerged in liquid: the top surfaces of the object is subject to less force due to pressure than the bottom surfaces.

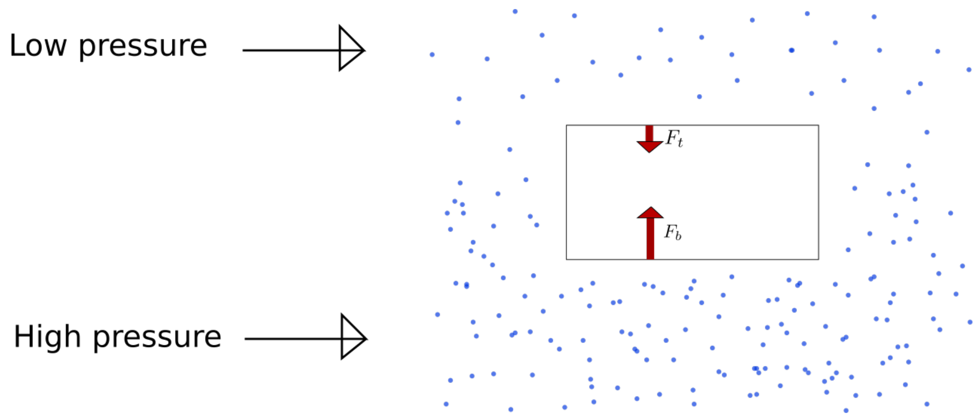


FIGURE 2.2. The difference of forces on the top and bottom surfaces is the resultant buoyancy force.

Task 1. Let $P(x, y, z)$ be the pressure of a liquid at the point (x, y, z). Suppose a ball of radius R centered at the origin is submerged in this liquid. Write down a double integral expression which computes the vertical component of the buoyancy force experienced by the ball. Justify your answer. [2]

The true function $P(x, y, z)$ might be quite involved. We will do computations under a simplified assumption. If we are interested in the pressure near some height z_0, under the assumption that the density of the liquid does not vary significantly near the values of z we are considering, the pressure is given by

$$P(x, y, z) = P_0 - \rho_0 g(z - z_0) \tag{1}$$

where g is the acceleration due to gravity, P_0 is the pressure at $z = z_0$ and ρ_0 is the density of air at $z = z_0$. Certainly this formula cannot be correct for all values of z: for one, it is eventually negative. To fix this, we would have to incorporate the fact that the air density also depends on z. Nonetheless, this approximation is sufficient as long as $z - z_0$ is relatively small, which will be sufficient for our purposes.

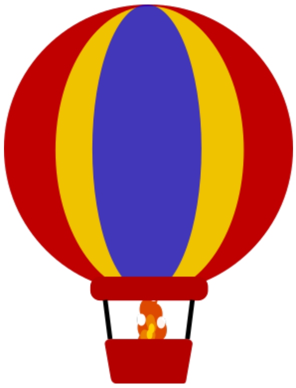
Task 2. Suppose a ball of radius R is suspended in the air at height z_0. Using your answer to task 1 and equation 1 for P, find the buoyancy force experienced by the ball in terms of R, ρ_0, and P_0. Evaluate the double integral directly without applying any integral theorems covered in the last two weeks of the semester.

3. Hot-air Balloon

We will use the computation in task 2 in order to determine the size of hot-air balloon we will need in order to lift a specified number of people.

[1] In the case of constant pressure, this sum vanishes. Can you think of a physical argument for why it is so?

[2] You do not need to write a proof, but you should have a convincing argument for why your expression is correct.



It might be tempting to think that the hot air inside of a hot-air balloon is what produces the lifting force on the balloon. In fact, the balloon experiences the same buoyancy force regardless of what gas it is filled with. The reason to fill the balloon with hot air is so that it is light enough for the buoyancy force to be able to lift the balloon off the ground.

A hot-air balloon consists of the balloon and the basket which carries people and the equipment. The total weight of the balloon therefore consists of the weight of the balloon, the weight of the hot air inside of the balloon, the weight of the basket with the fuel, and the weight of the passengers.

Task 3. Choose a positive natural number n and a temperature T between 80 °C and 120 °C. Find the minimum radius of the balloon which can hover at "sea level in standard atmospheric conditions"[3] with n people on board while heating the air to temperature T. [4]

You will need to look up some constants while I will give suggestions for others. (beware of the units). For the dependence of density on temperature, use the formula

$$\rho = \frac{P}{R_{specific}T}$$

where $R_{specific} = 287.058J/kgK$ and P is the pressure in standard atmospheric conditions.[5] For the weight of the basket with equipment you can use

$$m_b = 250kg.$$

For the density of the balloon material you can use

$$\eta_n = 64g/m^2$$

which should be roughly the density of nylon sheets used in some balloons.[6]

4. Archimedes' principle

In this section we will prove Archimedes' principle under the assumption on the pressure function we used at the end of section 2.

[3]Look up the pressure and temperature at sea level in standard atmospheric conditions.

[4]Of course if you were making a balloon you would make it slightly bigger so that it can also ascend.

[5]Temperature measured in Kelvin.

[6]You will need to find a root of a cubic equation. I suggest that you use Newton's method to find an approximation, but you may use a graphing tool instead.

Task 4. Find vector fields $\boldsymbol{H}_x, \boldsymbol{H}_y, \boldsymbol{H}_z$ such that for a body W immersed in a liquid of density ρ_0 at height z_0 the total buoyancy force $\boldsymbol{F} = \langle F_x, F_y, F_z \rangle$ exerted on W is given by surface integrals

$$F_x = \iint_{\partial W} \boldsymbol{H}_x \cdot d\boldsymbol{S}$$
$$F_y = \iint_{\partial W} \boldsymbol{H}_y \cdot d\boldsymbol{S}$$
$$F_z = \iint_{\partial W} \boldsymbol{H}_z \cdot d\boldsymbol{S}.$$

Use equation 1 for the pressure function.

Task 5. Look up the statement of the Archimedes' principle and prove it using the vector fields of task 4 and the divergence theorem.