Metric Spaces Worksheet 7

Topology III

Now we are ready to address the elephant in the room. There is indeed a relationship between the closed sets and the open sets in a metric space. In order to address, however, we must first establish a useful theorem about closed sets.

Theorem 1 (points outside a closed set are separated from that closed set)**.** *Let (X, d) be a metric space, $G \subseteq X$ be a closed set, and $x \in X \setminus G$ be a point outside G. There exists an $\varepsilon \in (0, \infty)$ such that $B_\varepsilon(x) \cap G = \emptyset$.*

Hint 2. To prove this aim for a contradiction,

1. Suppose this wasn't true, and understand what that means.
2. Argue that for every $n \in \mathbb{N}$, under this assumption there must be at least one point in $B_{\frac{1}{n+1}}(x) \cap G$.
3. By appealing to the ☠ Axiom of Choice ☠, define a sequence a_n by requiring that each $a_n \in B_{\frac{1}{n+1}}(x) \cap G$. (Essentially, you may assume there is such a sequence by invoking this plot device.)
4. Prove that this sequence converges.

Complete the proof here

(continued on next page)

Proof continued

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Theorem 3 (open iff complement is closed)**.** *In a metric space* (X, d)*, a subset* $U \subseteq X$ *is open iff its complement* U^c *is closed, where* $U^c :\equiv X \setminus U$*.*

To prove theorem 3, we can break up this statement into two parts.

Proposition 4 (complements of open sets are closed)**.** *In a metric space* (X, d)*, if* $U \subseteq X$ *is open then its complement* U^c *is closed.*

Hint 5. Look back at your proof of theorem 7 of Script 6, and try to figure out how to present a similar argument in the setting of a general metric space.

Complete the proof here

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Proposition 6 (complements of closed sets are open)**.** *In a metric space (X, d), if $G \subseteq X$ is closed then its complement G^c is open.*

Hint 7. The following proof skeleton may be useful:

1. Given a closed set G, to show that G^c is open, we must choose a point $x \in G^c$ and find an $\varepsilon \in (0, \infty)$ for which $B_\varepsilon(x) \subseteq G^c$.
2. Argue that $B_\varepsilon(x) \subseteq G^c \leftrightarrow B_\varepsilon(x) \cap G = \emptyset$.
3. Apply a previous result.

Complete the proof here

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We are now in a position to prove theorem 3 by combining the proofs of propositions 4 and 6.

Complete the proof of theorem 3

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Now that we understand the relationship between closed sets and open sets, it might seem natural to ask whether there are sets which are *both* open and closed. Whence we arrive at the following amusing terminology.

Definition 8 (clopen set)**.** A subset $S \subseteq X$ of a metric space (X, d) is said to be *clopen* if it is both open and closed. ⌟

Example 9 (*singletons are clopen in a discrete space*)

Let (X, d) be a discrete metric space, and $x \in X$ a point. The singleton set $\{x\}$ is clopen.

Hint 10. In proving that a set C is clopen, we may prove any one of:

1. C is both open and closed, directly
2. C is open and C^c is open
3. C is closed and C^c is closed
4. C^c is both open and closed, directly

Only context and experience can aid us in determining which route is likely to be easier.

Complete the proof here

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Question 11. Can you find a metric space in which every subset is clopen? If so, describe it mathematically. If not, prove that such a space cannot exist.

Complete your answer here

Review 12 (open sets in the Euclidean space $\mathbb{R}$)**.** Determine whether each of the following subsets of the Euclidean metric space $\mathbb{R}$ are open or closed or both or neither.

1. The interval $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ for fixed $a < b \in \mathbb{R}$.
2. The interval $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ for fixed $a < b \in \mathbb{R}$.
3. The interval $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$ for fixed $a < b \in \mathbb{R}$.
4. The interval $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$ for fixed $a \in \mathbb{R}$.
5. The interval $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$ for fixed $a \in \mathbb{R}$.
6. The point $\{0\} \in \mathbb{R}$.
7. The set $\mathbb{Z} \in \mathbb{R}$.
8. The set $\mathbb{Q} \in \mathbb{R}$.

Complete the review here