Metric Spaces Worksheet 6

Topology II

Last time we learned about *open* sets in metric spaces, whose study naturally leads to the complementary notion of *closed* sets.

Definition 1 (closed set)**.** A subset $G \subseteq X$ of a metric space (X, d) is called *closed* if, whenever (a_n) is a sequence in G and (a_n) converges, $\lim a_n \in G$. ⌟

At this point it may seem that determining that a set is closed requires substantially more work than would be involved in determining whether it is open. In particular, for an open set we need only prescribe an open ball for each point in that set — which can typically be done by leveraging the geometry of the set — but for a closed set we must argue about every convergent sequence — of which there might be uncountably many. Nevertheless, there are somehow 'degenerate' cases in which these arguments are simplified.

Example 2 (*examples of closed sets*)

Let (X, d) be a metric space. For each listed set below, show that it is closed.

1. The entire space, X, as well as the empty set $\emptyset$
2. $\{x\} \subseteq X$, for every point $x \in X$

Complete the proof here

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Our previous explorations of convergent sequences in discrete metric spaces can be leveraged to prove the following result.

Lemma 3 (discrete metric spaces)**.** *If (X, d) is a discrete metric space, then every subset $Y \subseteq X$ is closed.*

Complete the proof here

$\blacksquare$

Another important class of closed sets are the finite sets.

Lemma 4 (finite sets are always closed)**.** *If (X, d) is a metric space and $F \subseteq X$ is a finite subset of X then F is closed.*

Hint 5. Think back to the last time we considered sequences in finite metric spaces. What does a convergent sequence in a finite metric space look like?

Complete the proof here

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We now aim to prove that a large class of subsets of the Euclidean space $\mathbb{R}$ are actually closed and in so doing aim to obtain an intuitive idea about the 'shape' of closed sets. This class is that of the *closed intervals*. For any $a, b \in \mathbb{R}$ with $a \leq b$, we write the closed interval from a to b as

$$[a,b] :\equiv \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

Lemma 6 (distinct points are separated)**.** *Given two points $x, y \in X$ where (X, d) is a metric space, if $x \neq y$ then there exists $\varepsilon \in (0, \infty)$ such that $d(x, y) \geq \varepsilon$.*

Complete the proof here

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Theorem 7 (closed intervals are closed sets)**.** *In the Euclidean metric space $\mathbb{R}$, every closed interval is a closed set.*

Hint 8. What would happen if $\lim a_n \notin [a, b]$? Argue by cases on $\lim a_n > b$ and $\lim a_n < a$, recalling the definition of convergence.

Complete the proof here

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Much like in the case of open sets, closed sets may be combined in several ways such that the result is another closed set.

Theorem 9 (closed set laws)**.** *In a metric space* (X, d)*,*

1. X *and* $\emptyset$ *are closed sets.*
2. *If* $\mathcal{G}$ *is a family of closed sets then* $\cap_{G \in \mathcal{G}} G$ *is closed.*
3. *If* $G, H \subseteq X$ *are closed sets then* $G \cup H$ *is closed.*

Complete the proof here

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Surprise 10 (*union of closeds is not generally closed*)

In general, arbitrary unions of closed sets needn't be closed. In the Euclidean metric space $\mathbb{R}$ consider the following family of sets,

$$\mathcal{G} :\equiv \{[0, n/(1+n)] \mid n \in \mathbb{N}\} = \{[0, 1/2], [0, 2/3], \ldots\}$$

Show that the union of $\mathcal{G}$ is not closed by computing it and giving a sequence in it which converges, but whose limit is outside the set.

Complete the proof here

$\blacksquare$

With theorem 9 we can re-prove lemma 4, but this time we may argue by induction on the number of points in F.

Lemma 11 (finite sets are closed, again)**.** *If (X, d) is a metric space and $F \subseteq X$ is a finite subset of X then F is closed.*

Complete the proof here

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